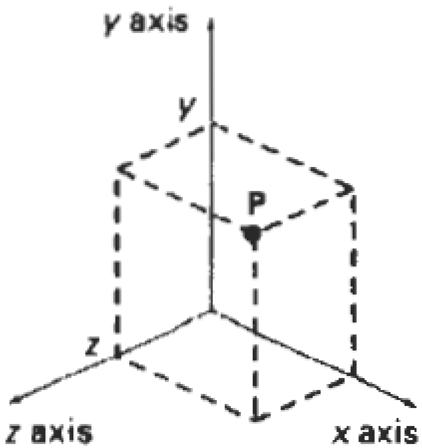
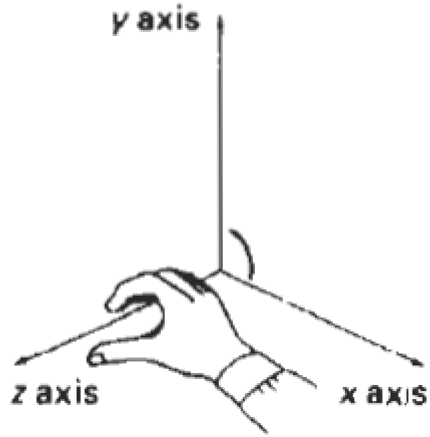
Introduction

When we model and display a *three-dimensional* scene, there are many more considerations we must take into account besides just including coordinate values for the third dimension. Object boundaries can be constructed with various combinations of plane and curved surfaces, and we sometimes we need to specify information about object interiors. Also, some geometric transformations are more involved in three-dimensional space than in two dimensions. For example, we can rotate an object about an axis with any spatial orientation in three-dimensional space. Two-dimensional rotations, on the other hand, are always around an axis that is perpendicular to the *xy* plane. Viewing transformations in three dimensions are much more complicated because we have many more parameters to select when specifying how a three-dimensional scene is to be mapped to a display device. The scene description must be processed through viewing-coordinate transformations and projection routines that transform three-dimensional viewing coordinates onto two-dimensional device coordinates. Visible parts of a scene, for a selected view, must be identified.

Coordinate System

A three dimensional coordinate system can be viewed as an extension of two dimensional coordinate system. the third dimension depth is represented by the Z-axis which is at right angle to the x,y coordinate plane. Figure below shows the conventional orientation for the coordinate axes in a three-dimensional Cartesian reference system. This is called a right-handed system because the right-hand thumb points in the positive z direction when we imagine grasping the z axis with the fingers curling from the positive x axis to the . positive y axis (through 90°). Most computer graphics packages require object descriptions and manipulations to be specified in right-handed Cartesian coordinates. For discussions throughout this course, we assume that all Cartesian reference frames are right-handed.





Another possible arrangement of Cartesian axes is the left-handed system. For the system, the left-hand thumb points in the positive z direction when we imagine grasping the z axis so that the fingers of the left hand curl from the positive x axis to the positive y axis through 90°. This orientation of axes is sometimes convenient for describing depth of objects relative to a display screen. If screen locations are described in the xy plane of a left-handed system with the coordinate origin in the lower-left screen corner, positive z values indicate positions behind the screen.

Transformations

Methods for geometric transformations in three dimensions are extended from two-dimensional methods by including considerations for the z coordinate. We now translate an object by specifying a three-dimensional translation vector, which determines how much the object is to be moved in each of the three coordinate directions. Similarly, we scale an object with three coordinate scaling factors. The extension for three-dimensional rotation is less straightforward. When we discussed two-dimensional rotations in the *xy* plane, we needed to consider only rotations about axes that were perpendicular to the *xy* plane. In three-dimensional space, we can now select any spatial orientation for the rotation axis. Most graphics packages handle three-dimensional rotation as a composite of three rotations, one for each of the three Cartesian axes. Alternatively, a user can easily set up a general rotation matrix, given the orientation of the axis and the required rotation angle. As in the two-dimensional case, we express geometric transformations in matrix form. Any sequence of transformations is then represented as a single matrix, formed by concatenating the matrices for the individual transformations in the sequence.

1. Translation (Moving)

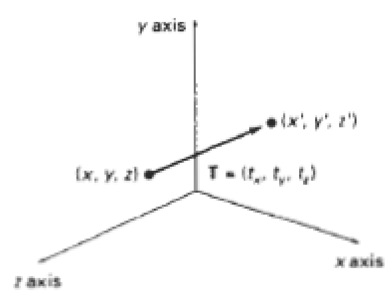
In a three-dimensional homogeneous coordinate representation, a point is translated from position P = (x, y, z) to position P' = (x', y', z') with the matrix operation:

$$[x \quad y \quad z \quad 1].\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix} = [\acute{x} \quad \acute{y} \quad \acute{z} \quad 1]$$

P.T= P'

Parameters t_x, t_y, and t_z specifying translation distances for the coordinate directions x, y, and z, are assigned any real values. The matrix representation is equivalent to the three equations

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$

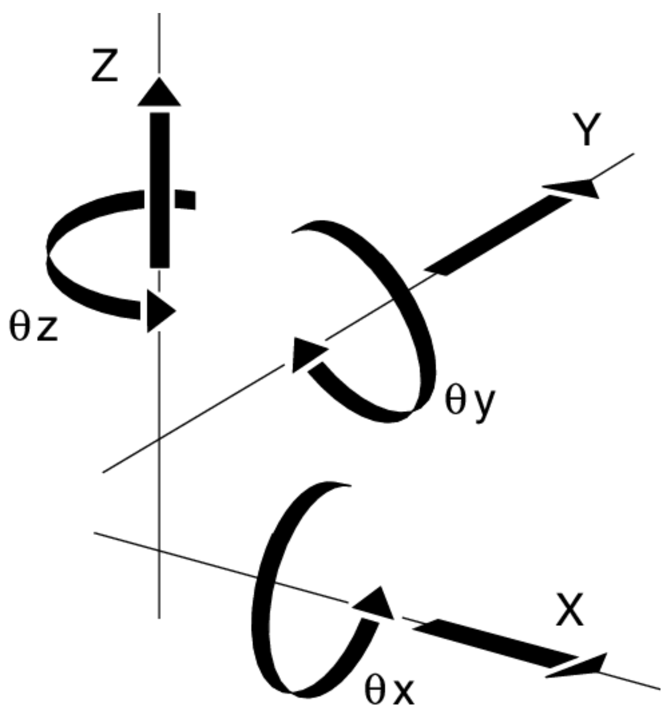


An object is translated in three dimensions by transforming each of the defining points of the object. For an object represented as a set of polygon surfaces, we translate each vertex of each surface and redraw the polygon facets in the new position.

We obtain the inverse of the translation matrix by negating the translation distances t_x, t_y, and t_z. This produces a translation in the opposite direction, and the product of a translation matrix and its inverse produces the identity matrix.

2. Rotation

To generate a rotation transformation for an object, we must designate an axis of rotation (about which the object is to be rotated) and the amount of angular rotation. Unlike two-dimensional applications, where all transformations are carried out in the xy plane, a three-dimensional rotation can be specified around any line in space. The easiest rotation axes to handle are those that are parallel to the coordinate axes. Also, we can use combinations of coordinate axis rotations (along with appropriate translations) to specify any general rotation.



By convention, positive rotation angles produce counterclockwise rotations about a coordinate axis, if we are looking along the positive half of the axis toward the coordinate origin. This agrees with the rotation in two dimensions, where positive rotations in the *xy* plane are counterclockwise about axes parallel to the z axis.

A. Coordinate-Axes Rotations

The two-dimensional z-axis rotation equations are easily extended to three dimensions:

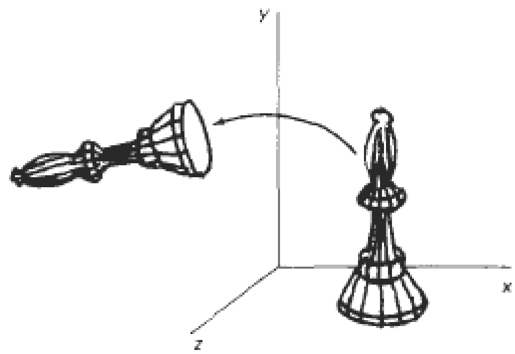
$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$z' = z$$

Parameter θ specifies the rotation angle. In homogeneous coordinate form, the three-dimensional z-axis rotation equations are expressed as:

$$[\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \quad \mathbf{1}].\begin{bmatrix} \boldsymbol{cos\,\theta} & \boldsymbol{sin\,\theta} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{sin\,\theta} & \boldsymbol{cos\,\theta} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = [\boldsymbol{\acute{x}} \quad \boldsymbol{\acute{y}} \quad \boldsymbol{\acute{z}} \quad \mathbf{1}]$$



which we can write more compactly as:

$$P.R_z(\theta) = P'$$

Transformation equations for rotations about the other two coordinate axes can be obtained with a cyclic permutation of the coordinate parameters x, y, and z. That is, we use the replacements:

$$x \longrightarrow y \longrightarrow z \longrightarrow x$$

Substituting permutations, we get the equations for an x-axis rotation:

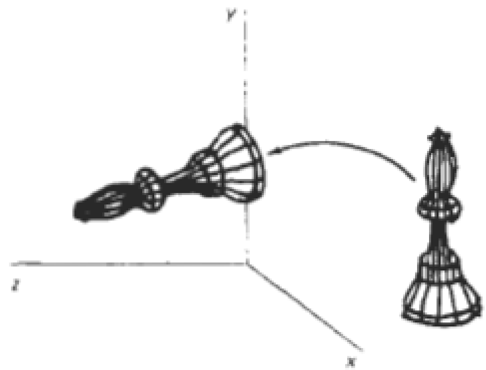
$$x' = x$$

$$y' = y\cos\theta - z\sin\theta$$

$$z' = y\sin\theta + z\cos\theta$$

which can be written in the homogeneous coordinates form

$$[\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \quad \mathbf{1}].\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\cos\theta} & \boldsymbol{\sin\theta} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sin\theta} & \boldsymbol{\cos\theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = [\boldsymbol{\acute{x}} \quad \boldsymbol{\acute{y}} \quad \boldsymbol{\acute{z}} \quad \mathbf{1}]$$



which we can write more compactly as:

$$P.R_x(\theta) = P'$$

Cyclically permuting coordinates give us the transformation equations for a y-axis rotation

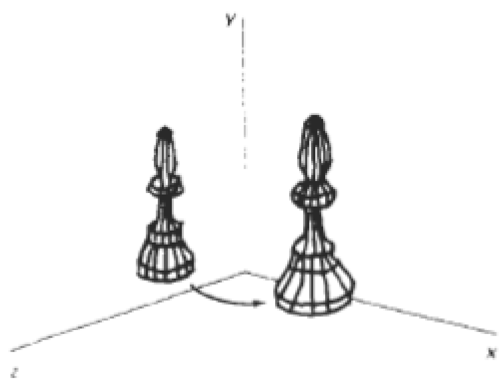
$$x' = z\sin\theta + x\cos\theta$$

$$y' = y$$

$$z' = z\cos\theta - x\sin\theta$$

which can be written in the homogeneous coordinates form

$$[\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \quad \mathbf{1}].\begin{bmatrix} \boldsymbol{cos\,\theta} & \mathbf{0} & \boldsymbol{-sin\,\theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{sin\,\theta} & \mathbf{0} & \boldsymbol{cos\,\theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = [\boldsymbol{\acute{x}} \quad \boldsymbol{\acute{y}} \quad \boldsymbol{\acute{z}} \quad \mathbf{1}]$$



which we can write more compactly as:

$$P.R_y(\theta) = P'$$

An inverse rotation matrix is formed by replacing the rotation angle θ by - θ. Negative values for rotation angles generate rotations in a clockwise direction, so the identity matrix is produced when any rotation matrix is multiplied by its inverse. Since only the sine function is affected by the change in sign of the rotation angle, the inverse matrix can also be obtained by interchanging rows and columns. That is, we can calculate the inverse of any rotation matrix R by evaluating its transpose ($R^{-I} = R^T$). This method for obtaining an inverse matrix holds also for any composite rotation matrix.