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## 3.Normed Spaces

### 3.1 Definitions and General Properties

Let $X$ be a linear space over $F$ where $F$ is either the field of real numbers or the field of complex numbers.

## Definition(3.1.1)

A norm on $X$ is a function $\|\|:. X \rightarrow \mathbb{R}$ having the following properties.
(1) $\|x\| \geq 0$ for all $x \in X$
(2) $\|x\|=0$ iff $x=0$
(3) $|\lambda x\|=|\lambda| \mid x\|$ for all $x \in X$ and for all $\lambda \in F$
(4) $\|x+y\| \leq\|x\|+\|y\|$ for all ${ }^{\circ}, y \in X$

The linear $X$ over $F$ together with $\|$.$\| is called a normed space and is denoted by (X,\|\|$.$) or$ simply $X$.

## Remark:

Every subspace of normed space is also normed space.

## Definition(3.1.2)

A seminorm on $X$ is a function $p: X \rightarrow \mathbb{R}$ having the following
(1) $p(\lambda x)=|\lambda| p(x)$ for all $x \in X$ and for all $\lambda \in F$ (2) $p(x+y) \leq p(x)+p(y)$ for all $x, y \in X$

A family $F$ of seminorms on is said to be separating if to each $x \neq 0$ corresponds at least one $p \in F$ with $p(x) \neq 0$.

## Theorem(3.1.3)

Suppose $p$ is a seminorm on a vector space $X$. Then
(1) $p(0)=0$
(2) $p(-x)=p(x)$ for all $x \in X$
(3) $p(y-x)=p(x-y)$ for all $x, y \in X$
(4) $|p(x)-p(y)| \leq p(x-y)$ for all $x, y \in X$
(5) $p(x) \geq 0$ for all $x \in X$
(6) The set $N(p)=\{x \in X \circ p(x)=0\}$ is a subspace of $X$
(7) The set $A=\{x \in X ; \mathscr{p}(x)<1\}$ is convex, balanced and absorbing set.
(8) $p$ is a norm if it satisfies the condition $p(x) \neq 0$ if $x \neq 0$

## Proof :

(1), (2) and (3) direct from definition
(4) $x=(x-y)+y$

$$
\begin{equation*}
\Rightarrow \quad p(x)=p((x-y)+y) \leq p(x-y)+p(y) \tag{2}
\end{equation*}
$$

$p(x)-p(y) \leq p(x-y)$
$\cdots$ (1) Also $-p(x-y) \leq p(x)-p(y)$
From(1) and (2), we nave
$-p(x-y) \leq p(x)-p(y) \leq p(x-y) \Rightarrow|p(x)-p(y)| \leq p(x-y)$
(5) Since $|p(x)-p(y)| \leq p(x-y)$ for all $x, y \in X$

Take $y=0 \Rightarrow|p(x)| \leq p(x)$
Since $|p(x)| \geq 0 \Rightarrow p(x) \geq 0 \quad$ for all $x \in X$

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(6) Since $p(0)=0 \Rightarrow 0 \in N(p) \Rightarrow N(p) \neq \phi$

Let $x, y \in N(p)$ and $\alpha, \beta \in F \Rightarrow p(x)=0, \quad P(y)=0$
$p(\alpha x+\beta y) \leq p(\alpha x)+p(\beta y) \leq|\alpha| p(x)+|y| p(y)=0 \quad \Rightarrow \quad p(\alpha x+\beta y) \leq 0$
Since $x, y \in N(p) x, y \in N(p), \alpha, \beta \in F$, and $X$ is vector space, then $\alpha x+\beta y \in X \quad \Rightarrow \quad p(\alpha x+\beta y) \geq 0$ $p(\alpha x+\beta y)=0 \Rightarrow \alpha x+\beta y \in N(p) \Rightarrow N(p)$ is a subspace.
(a) Let $x, y \in A$ and $0 \leq \lambda \leq 1$, then $p(x)<1, \quad p(y)<1$
$p(\lambda x+(1-\lambda) y) \leq p(\lambda x)+p((1-\lambda) y)=|\lambda| p(x)+|1-\lambda| p(y)=\lambda p(x)+(1-\lambda) p(y)$
Since $p(x)<1, p(y)<1 \Rightarrow \lambda p(x)<1, \quad(1-\lambda) p(y)<1-\lambda$
$\lambda p(x)+(1-\lambda) p(y)<\lambda+(1-\lambda)=1 \Rightarrow p(\lambda x+(1-\lambda) y)<1$
$\lambda x+(1-\lambda) y \in A \quad \Rightarrow A$ is convex
(b) Let $\lambda \in F$ with $|\lambda| \leq 1$

Let $x \in \lambda A \Rightarrow x=\lambda y$ where $y \in A \quad \Rightarrow \quad p(y)<1$
Since $p(x)=p(\lambda y)=|\lambda| p(y)$ and $|\lambda| \leq 1, p(y)<1$
$\Rightarrow|\lambda| p(y)<1 \Rightarrow p(x)<1 \Rightarrow x \in A \Rightarrow \lambda A \subset A \Rightarrow A$ is balanced
(c) Let $x \in X$ and let $p(x)<\lambda \Rightarrow \lambda>0 \Rightarrow p\left(\lambda^{-1} x\right)<1 \Rightarrow \lambda^{-1} x \in A \Rightarrow x \in \lambda A$
$\Rightarrow A$ is absorbing.

## Theorem(3.1.4)

Every normed space is metric space

## Proof :

Let $(X,\|$.$\| ) be a normed space: Define d: X \times X \rightarrow R$ by $d(x, y)=\|x-y\|$ for all $x, y \in X$
(1)Let $x, y \in X \Rightarrow x-y \in X$ (because $X$ is a vector space) $\Rightarrow\|x-y\| \geq 0 \Rightarrow d(x, y) \geq 0$
(2) Let $x, y \in X$

$$
d(x, y)=0 \Leftrightarrow\|x-y\|=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y
$$

(3) Let $x, y \in X \Rightarrow d(x, y)=\|x-y\|=\|y-x\|=d(y, x)$
(4)Let $x, y, z \in X$

$$
\|x-y\|=\|(x-z)+(z-y)\| \leq\|x-z\|+\|z-y\| \Rightarrow d(x, y) \leq d(x, z)+d(z, y)
$$

It follows that $d$ is a metric on $X$, and this metric is called the metric induced by the normed.

## Remark:

If $x, y, z \in X$ and $\lambda \in F$, then
(1) $d(x+z, y+z)=d(x, y)$
(2) $d(\lambda x, \lambda y)=|\lambda| d(x, y)$
(3) $\|x\|=d(x, 0)$

## Definition(3.1.5)

A topological linear space $X$ is said to be normable if a norm exists on $X$ such that the metric induced by the norm is compatible with $\tau$.

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## Definition(3.1.6)

Let $X$ be normed space. The open ball with center $x_{0}$ and radius $r$ is dented by $\beta_{r}\left(x_{0}\right)$ and defined as $\beta_{r}\left(x_{0}\right)=\left\{x \in X:\left\|x-x_{0}\right\|<r\right\}$, and the closed ball is $\overline{\beta_{r}}\left(x_{0}\right)=\left\{x \in X:\left\|x-x_{0}\right\| \leq r\right\}$

## Remark

$\beta_{r}\left(x_{0}\right)=x_{0}+\beta_{r}(0)=x_{0}+r \beta_{1}(0)$
Indeed
$\beta_{r}\left(x_{0}\right)=\left\{x \in X:\left\|x-x_{0}\right\|<r\right\}=\left\{x_{0}+y:\|y\|<r\right\}=x_{0}+\{y:\|y\|<r\}=x_{0}+\beta_{r}(0)$
Also $\beta_{r}(0)=\{x \in X:\|x\|<r\}=\left\{x \in X: \frac{\|x\|}{r}<1\right\}=\{r y:\|y\|<1\}=r\{y:\|y\|<1\}=r \beta_{1}(0)$

## Example (3.1.7)

Every open and closed balls in normed space are convex

## Ans:

Let $X$ be normed space
Let $x, y \in B_{r}\left(x_{0}\right)$ and $0 \leq \lambda \leq 1 \Rightarrow\left\|x-x_{0}\right\|<r, \quad\left\|y-x_{0}\right\| \leq r$ We must to prove $\lambda x+(1-\lambda) y \in B_{r}\left(x_{0}\right)$ $\lambda x+(1-\lambda) y-x_{0}=\lambda\left(x-x_{0}\right)+(1-\lambda)\left(y-x_{0}\right)$
$\left\|\lambda x+(1-\lambda) y-x_{0}\right\|=\left\|\lambda\left(x-x_{0}\right)+(1-\lambda)\left(y-x_{0}\right)\right\| \leq|\lambda|\left\|x-x_{0}\right\|+|1-\lambda|\left\|y-x_{0}\right\|<\lambda r+(1-\lambda) r=r$
since $|1-\lambda|=1-\lambda,|\lambda|=\lambda$, because $\lambda, 1-\lambda \geq 0$
$\Rightarrow \lambda x+(1-\lambda) y \in B_{r}\left(x_{0}\right) \Rightarrow B_{r}\left(x_{0}\right)$ is convex. Similarly, to prove $\overline{B_{r}}\left(x_{0}\right)$ is convex.

## Equivalent Norms

## Definition(3.1.8)

Let $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ be two norms on vector space $X$.We say that $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ are equivalent (or $\|\cdot\|_{1}$ is equivalent to $\|\cdot\|_{2}$ ), written $\|\cdot\|_{1} \sim\|\cdot\|_{2}$, if there exists positive real numbers $a$ and $b$ such that $a\|x\|_{1} \leq\|x\|_{2} \leq b\|x\|_{1}$ for all $x \in X$.

## Example(3.1.9)

Let $\|x\|_{1}=\sum_{i=1}^{n} \mid x_{i}{ }^{\prime}$ and $\|x\|_{2}=\left(\sum_{i=1}^{n} x_{i}^{2}\right)^{\frac{1}{2}}$ for all $x=\left(x_{1}, \ldots, x_{n}\right) \in R^{n}$. Show that $\|\cdot\|_{1} \sim\|\cdot\|_{2}$
Ans:
From Cauchy's inequality, we have $\sum_{i=1}^{n}\left|x_{i} y_{i}\right| \leq\left(\sum_{i=1}^{n} x_{i}^{2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} y_{i}^{2}\right)^{\frac{1}{2}}$ for all $x_{i}, y_{i} \in \mathbb{R}$
Put $y_{i}=1$ for all $i=1, \ldots n$, we have $\sum_{i=1}^{n}\left|x_{i}\right| \leq\left(\sum_{i=1}^{n} x_{i}^{2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} 1\right)^{\frac{1}{2}}$

$$
\|x\|_{1} \leq\|x\|_{2} \cdot \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}}\|x\|_{1} \leq\|x\|_{2} \Rightarrow a=\frac{1}{\sqrt{n}} \text {, but }\|x\|_{2} \leq\|x\|_{1} \Rightarrow b=1 \text {. Hence }\|\cdot\|_{1} \sim\|\cdot\|_{2}
$$

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## Lemma(3.1.10) Linear combination

Let $\left\{x_{1}, \ldots, x_{n}\right\}$ be a linear independent set of vectors in a normed space $X$, then there is a number $c>0$ such that $\left\|\sum_{i=1}^{n} \lambda_{i} x_{i}\right\| \geq c \sum_{i=1}^{n}\left|\lambda_{i}\right|$ for all $\lambda_{i} \in F, i=1, . ., n$

## Theorem(3.1.11)

On a finite dimensional vector space all norms are equivalent.

## Proof :

Let $X$ be finite dimensional vector space with $\operatorname{dim} X=n>0$, and $\|\cdot\|_{1},\|\cdot\|_{2}$ be two norms on $X$.
To prove $\|\cdot\|_{1} \sim\|\cdot\|_{2}$
Let $\left\{x_{1}, \ldots, x_{n}\right\}$ be a basis for $X \Rightarrow$ every $x \in X$ has a unique representation

$$
\begin{equation*}
x=\sum_{i=1}^{n} \lambda_{i} x_{i}, \quad \lambda_{i} \in F \quad \cdots(1) \text { and } \quad\|x\|_{1}=\left\|\sum_{i=1}^{n} \lambda_{i} x_{i}\right\|_{1} \leq \sum_{i=1}^{n} \mid \lambda_{i}\| \| x_{i} \|_{1} \tag{2}
\end{equation*}
$$

Put $k=\max \left\{\left\|x_{1}\right\|_{1}, \ldots,\left\|x_{n}\right\|_{1}\right\} \Rightarrow k$ for all $i=1, \ldots, n \Rightarrow \sum_{i=1}^{n}\left|\lambda_{i}\| \|_{i} \|_{1} \leq k \sum_{i=1}^{n}\right| \lambda_{i} \mid$
From (2) and (3), we have $\|x\|_{1} \leq k \sum_{i=1}^{n}\left|\lambda_{i}\right|$
Since the set $\left\{x_{1}, \ldots, x_{n}\right\}$ is linear independent, by lemma of linear independent, there is $c>0$ such that $\quad\left\|\sum_{i=1}^{n} \lambda_{i} x_{i}\right\|_{2} \geq c \sum_{i=1}^{n}\left|\lambda_{i}\right|$
From (1) and (5), we have $\quad\|x\|_{2} \geq c \sum_{i=1}^{n}\left|\lambda_{i}\right|$
From (4) and (6), we have $\|x\|_{1} \leq \frac{k}{c}\|x\|_{2}$
Put $a=\frac{c}{k}$, we have $\quad \rho^{a\|x\|_{1} \leq\|x\|_{2}}$
Similarly $\|x\|_{2} \subset k \sum_{i=1}^{n}\left|\lambda_{i}\right| \quad \cdots(8)$ and $\|x\|_{1} \geq c \sum_{i=1}^{n}\left|\lambda_{i}\right|$
From (8) and (9), we have $\|x\|_{2} \leq \frac{k}{c}\|x\|_{1}$
Put $b=\frac{k}{c}$, we have $\|x\|_{2} \leq b\|x\|_{1}$
From (7) and (10), we have $a\|x\|_{1} \leq\|x\|_{2} \leq b\|x\|_{1}$. Hence $\|\cdot\|_{1} \sim\|\cdot\|_{2}$

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### 3.2 Banach Spaces

## Definition(3.2.1)

A sequence $\left\{x_{n}\right\}$ in normed space $X$ is said to be
(1) Bounded in $X$, if there exists $M>0$ such that $\left\|x_{n}\right\| \leq M$ for all $n \in \mathbb{Z}^{+}$
(2) converge to a point $x \in X$ if, for every $\varepsilon>0$, there exists $k \in Z^{+}$such that $\left\|x_{n}-x\right\|<\varepsilon$ for all $n \geq k$. We write $\lim _{n \rightarrow \infty} x_{n}=x$ or $x_{n} \rightarrow x$ as $n \rightarrow \infty$
It follows that $x_{n} \rightarrow x$ iff $\left\|x_{n}-x\right\| \rightarrow 0$
(3) converge in $X$ if there is a point $x \in X$ such that $x_{n} \rightarrow x$. Otherwise, the sequence is said to be divergent.
(4) Cauchy sequence in $X$, if for every $\varepsilon>0$, there exists $k \in \mathbb{Z}^{+}$such that $\left\|x_{n}-x_{m}\right\|<\varepsilon$ for all $n, m \geq k$
It is clear to show that in any normed space
(1) Every convergent sequence is a Cauchy sequence.
(2) Every Cauchy sequence is bounded.

## Theorem (3.2.2)

Let $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ be two sequences in a normed space $X$ such that $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$
(1) $x_{n}+y_{n} \rightarrow x+y$
(2) $\lambda x_{n} \rightarrow \lambda x$ for all $\lambda \in F$
(3) $\left\|x_{n}\right\| \rightarrow\|x\|$
(4) $\left\|x_{n}-y_{n}\right\| \rightarrow\|x-y\|$
(5) If $\left\{\lambda_{n}\right\}$ is a sequence in $F$ such that $\lambda_{n} \rightarrow \lambda$, then $\lambda_{n} x_{n} \rightarrow \lambda x$

## Proof :

(1) $\left\|\left(x_{n}+y_{n}\right)-(x+y)\right\|=\left\|\left(x_{n}-x\right)+\left(y_{n}-y\right)\right\| \leq\left\|x_{n}-x\right\|+\left\|y_{n}-y\right\|$

Since $\left\|x_{n}-x\right\| \rightarrow 0$ and $\left\|y_{n}-y\right\| \rightarrow 0$ as $n \rightarrow \infty$, then $\left\|\left(x_{n}+y_{n}\right)-(x+y)\right\| \rightarrow 0$ as $n \rightarrow \infty$, i.e. $x_{n}+y_{n} \rightarrow x+y$
(3) Since $\left|\left\|x_{n}\right\|-\|x\|\right| \leq\left\|x_{n}-x\right\|$ and $\left\|x_{n}-x\right\| \rightarrow 0$ as $n \rightarrow \infty$, then $\left|\left\|x_{n}\right\|-\|x\|\right| \rightarrow 0$ as $n \rightarrow \infty$, i.e. $\left\|x_{n}\right\| \rightarrow\|x\|$
(4) $\left|\left\|x_{n}-y_{n}\right\|-\|x-y\|\right| \leq\left\|\left(x_{n}-y_{n}\right)-(x-y)\right\| \leq\left\|x_{n}-x\right\|+\left\|y_{n}-y\right\|$

Since $\left\|x_{n}-x\right\|_{-} \rightarrow 0$ and $\left\|y_{n}-y\right\| \rightarrow 0$ as $n \rightarrow \infty$, then $\left|\left\|x_{n}-y_{n}\right\|-\|x-y\|\right| \rightarrow 0$ as $n \rightarrow \infty$
(5) $\left\|\lambda_{n} x_{n}-\lambda x\right\|=\left\|\lambda_{n} x_{n}-\lambda_{n} x+\lambda_{n} x-\lambda x\right\|=\left\|\lambda_{n}\left(x_{n}-x\right)+\left(\lambda_{n}-\lambda\right) x\right\| \leq\left|\lambda_{n}\left\|x_{n}-x\right\|+\right| \lambda_{n}-\lambda\|x\|$

Since $)\left|\mid x_{n}-x \| \rightarrow 0\right.$ and $| \lambda_{n}-\lambda \mid \rightarrow 0$ as $n \rightarrow \infty$, then $\left|\lambda_{n} x_{n}-\lambda x\right| \rightarrow 0$ as $n \rightarrow \infty$

## Definition(3.2.3)

A normed space $X$ is called complete if every Cauchy sequence in $X$ is converge to a point of $X$. A complete normed space is called a Banach space .

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## Example(3.2.4)

Let $X=F$, we define the function $\|\|:. X \rightarrow \mathbb{R}$ by $\|x\|=|x|$ for all $x \in F$. Show that $X$ is a Banach space.
Ans :
First, to show $X$ is a normed space.
(1) Since $|x| \geq 0$ for all $x \in F \Rightarrow\|x\| \geq 0$
(2) Let $x \in X \Rightarrow\|x\|=0 \Leftrightarrow|x|=0 \Leftrightarrow x=0$
(3) Let $x \in X$ and $\lambda \in F \Rightarrow|\lambda x||=|\lambda x|=|\lambda|| x|=| \lambda\|x\|$
(4) Let $x, y \in X \Rightarrow|x+y\|=|x+y| \leq|x|+|y|=\| x\|+\| y \|$
$\Rightarrow X$ is a normed space
Since $\mathbb{R}$ or $\mathbb{C}$ is complete space, then $F$ is complete space, hence, $X$ is a Banach space.

## Remark

Let $F^{n}$ denote the set of all ordered n-tupies of elements in $F$ of fixed $n \in \mathbb{N}$, i.e.
$F^{n}=\left\{x=\left(x_{1}, \ldots, x_{n}\right): x_{i} \in F, i=1,2,3, \ldots, n\right\}$. Then $F^{n}$ is a vector space under the following addition and scalar multiplication
(1) $x+y=\left(x_{1}, \ldots, x_{n}\right)+\left(y_{1}, \ldots, y_{n}\right)=\left(x_{1}+y_{1}, \ldots, x_{n+} y_{n}\right)$ for all $x, y \in F^{n}$
(2) $\lambda x=\lambda\left(x_{1}, \ldots, x_{n}\right)=\left(\lambda x_{1}, \ldots, \lambda x_{n}\right)$ for all $x \in F^{n}$ and for all $\lambda \in F$

## Example(3.2.5)

Let $X=F^{n}$, we define the function $\forall \|: X \rightarrow \mathbb{R}$ by $\|x\|=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{2}\right)^{\frac{1}{2}}$ for all $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in F^{n}$.
Show that $X$ is a Banach space.

## Ans :

First, to show $X$ is a normed space.
(1) Since $\left|x_{i}\right|^{2} \geq 0$ for all $i=1, \ldots, n \Rightarrow\|x\| \geq 0$
(2) Let $x \in X$

$$
\begin{aligned}
& \|x\|=0 \Leftrightarrow\left(\sum_{i=1}^{m}\left|x_{i}\right|^{2}\right)^{\frac{1}{2}}=0 \quad \Leftrightarrow \sum_{i=1}^{n} x_{i}^{2}=0 \quad \Leftrightarrow x_{i}^{2}=0 \quad \forall i=1,2, \cdots, n \\
& \|x\|=0 \quad \Leftrightarrow x_{i}=0 \quad \forall i=1,2,3, \cdots, n \quad \Leftrightarrow \quad x=0
\end{aligned}
$$

(3) Let $x \in X$ and $\lambda \in F \Rightarrow\|\lambda x\|=\left(\sum_{i=1}^{n}\left|\lambda x_{i}\right|^{2}\right)^{\frac{1}{2}}=|\lambda|\left(\sum_{i=1}^{n}\left|x_{i}\right|^{2}\right)^{\frac{1}{2}}=|\lambda|\|x\|$
(4) Let $x, y \in X \Rightarrow\|x+y\|=\left(\sum_{i=1}^{n}\left(x_{i}+y_{i}\right)^{2}\right)^{\frac{1}{2}}$

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By using Minkowski's inequality, we have $\|x+y\| \leq\left(\sum_{i=1}^{n} x_{i}^{2}\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n} y_{i}^{2}\right)^{\frac{1}{2}}=\|x\|+\|y\|$
$\Rightarrow X$ is a normed space
Second to show $X$ is a Banach space.
Let $\left\{x_{m}\right\}$ be a Cauchy sequence in $X \Rightarrow x_{m} \in F^{n}$, we shall write $x_{m}=\left(x_{i m}, \ldots, x_{n m}\right)$
Let $\varepsilon>0$, since $\left\{x_{m}\right\}$ is a Cauchy sequence in $X$, there is $k \in \mathbb{Z}^{+}$such that $\left\|x_{m}-x_{l}\right\|<\varepsilon$ for all $m, l \geq k$ $\left\|x_{m}-x_{l}\right\|^{2}<\varepsilon^{2} \quad$ for all $m, l \geq k$
$x_{m}-x_{l}=\left(x_{1 m}-x_{1 l}, \ldots, x_{n m}-x_{n l}\right) \Rightarrow\left\|x_{m}-x_{l}\right\|^{2}=\sum_{i=1}^{n}\left|x_{i m}-x_{i l}\right|^{2}$
From (1) and (2), we have $\sum_{i=1}^{n}\left|x_{i m}-x_{i l}\right|^{2}<\varepsilon^{2}$ for all $m, l \geq k$
$\left|x_{i m}-x_{i l}\right|^{2}<\varepsilon^{2} \quad$ for all $m, l \geq k \Rightarrow\left|x_{i m}-x_{i l}\right|<\varepsilon$ for all $m, l \geq k$
So that, for each $i$, the sequence $\left\{x_{i n}\right\}$ is Cauchy sequence in $F$
Since $F$ is complete, then for each $i$, the sequence $\left\{x_{i m}\right\}$ is converges to a point, say $x_{i} \in F$, then $x_{i m} \rightarrow x_{i}$ for all $i=1,2, \cdots, n$
Put $x=\left(x_{1}, \cdots, x_{n}\right) \Rightarrow x \in F^{n}$. we must prove $x_{m} \rightarrow x$
Let $\varepsilon>0$, for all $m>k$, we have $\left\|x_{m}-x\right\|^{2}=\sum_{i=1}^{n}\left|x_{i m}-x_{i}\right|^{2}<\varepsilon^{2} \Rightarrow\left\|x_{m}-x\right\|<\varepsilon$ for all $m>k$
$\Rightarrow\left\{x_{m}\right\}$ is converges $\Rightarrow X$ is complete space, so that $X$ is Banach space.

## Example(3.2.6)

Let $X=\mathbb{R}^{n}$, we define the function $\|\cdot\|: X \rightarrow \mathbb{R}$ by $\|x\|=\sum_{i=1}^{n}\left|x_{i}\right|$
for all $x=\left(x_{1}, x_{2}, \cdots, x_{n}\right) \in \mathbb{R}^{n}$. Show that $X$ is a Banach space.
Ans:
First, to show $X$ is a normed space.
(1) Since $\left|x_{i}\right| \geq 0$ for all $i=1, \cdots, n \quad \Rightarrow \quad\|x\| \geq 0$
(2) Let $x \in X$.

$$
\left|x \|=0 \Leftrightarrow \sum_{i=1}^{n}\right| x_{i}|=0 \Leftrightarrow| x_{i} \mid=0 \quad \forall i=1,2, \cdots, n \quad \Leftrightarrow x_{i}=0 \quad \forall i=1,2, \cdots, n \quad \Leftrightarrow x=0
$$

(3) Let $x \in X$ and $\lambda \in \mathbb{R}$

$$
\lambda x=\lambda\left(x_{1}, \cdots, x_{n}\right)=\left(\lambda x_{1}, \cdots, \lambda x_{n}\right) \Rightarrow\|\lambda x\|=\sum_{i=1}^{n}\left|\lambda x_{i}\right|=\sum_{i=1}^{n}|\lambda|\left|x_{i}\right|=|\lambda| \sum_{i=1}^{n}\left|x_{i}\right|=|\lambda| \| x_{i} \mid
$$

(4) Let $x, y \in X$

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$$
\|x+y\|=\sum_{i=1}^{n}\left|x_{i}+y_{i}\right|=\sum_{i=1}^{n}\left(\left|x_{i}\right|+\left|y_{i}\right|\right) \leq \sum_{i=1}^{n}\left|x_{1}\right|+\sum_{i=1}^{n}\left|y_{i}\right| \Rightarrow\|x+y\| \leq\|x\|+\|y\|
$$

$\Rightarrow X$ is a normed space
Second to show $X$ is a Banach space.

## Example(3.2.7)

Let $X=\mathbb{R}^{n}$, we define the function $\|\cdot\|: X \rightarrow \mathbb{R}$ by $\|x\|=\max \left\{\left|x_{1}\right|,\left|x_{2}\right|, \cdots,\left|x_{n}\right|\right\}$ for all $x=\left(x_{1}, x_{2}, \cdots, x_{n}\right) \in \mathbb{R}^{n}$. Show that $X$ is a Banach space.
Ans :
First, to show $X$ is a normed space.
(1) Since $\left|x_{i}\right| \geq 0$ for all $i=1, \cdots, n \Rightarrow\|x\| \geq 0$
(2) ) Let $x \in \mathbb{R}^{n}$

$$
\|x\|=0 \Leftrightarrow \max \left\{\left|x_{1}\right|, \cdots,\left|x_{n}\right|\right\}=0 \Leftrightarrow\left|x_{i}\right|=0 \quad \forall i=1,2, \cdots, n \Leftrightarrow x_{i} \mathcal{F}_{i} \theta \quad \forall i=1,2, \cdots, n \Leftrightarrow x=0
$$

(3) Let $x \in \mathbb{R}^{n}$ and $\lambda \in \mathbb{R}$

$$
\begin{aligned}
& \lambda x=\lambda\left(x_{1}, \cdots, x_{n}\right)=\left(\lambda x_{1}, \cdots, \lambda x_{n}\right) \\
& \| \lambda x\left|=\max \left\{\left|\lambda x_{1}\right|, \ldots,\left|\lambda x_{n}\right|\right\}=\max \left\{|\lambda| x_{1}\left|, \ldots,|\lambda| x_{n}\right|\right\}=|\lambda| \max \left\{\left|x_{1}, \ldots,\left|x_{n}\right|\right\}=|\lambda||x|\right.\right.
\end{aligned}
$$

(4) Let $x, y \in \mathbb{R}^{n}$
$|x+y|=\max \left\{\left|x_{1}+y_{1}\right|, \cdots,\left|x_{n}+y_{n}\right|\right\} \leq \max \left\{\left|x_{1}\right|+|y|, \cdots, \mid x_{n},+y_{n}\right\} \leq \max \left\{\left|x_{1}\right|, \cdots, \mid x_{n}\right\}+\max \left\{\left|y_{1}\right|, \cdots,\left|y_{n}\right|\right\}=\| x|+|y|$
$X$ is a normed space
Second to show $X$ is a Banach space.

## Remark

Let $C[a, b]$ denote the set of all real-valued bounded continuous functions defined on $[a, b]$, i.e. $f \in C[a, b]$ iff $f:[a, b] \rightarrow \mathbb{R}$ is bounded and continuous functions. Then $C[a, b]$ is a vector space under the following addition and scalar multiplication
(1) $(f+g)(x)=f(x)+g(x)$ for all $f, g \in C[a, b]$
(2) $(\lambda f)(x)=\lambda f(x)$ for all $f \in C[a, b]$ and for all $\lambda \in F$

## Example(3.2.8)

Let $X=C[a, b]$, we define the function $\|\cdot\|: X \rightarrow \mathbb{R}$ by $\|f\|=\max \{|f(x)|: a \leq x \leq b\}$ for all $f \in X$. Show that $X$ is a Banach space.
Ans:
First, to show $X$ is a normed space.
(1) Since $|f(x)| \geq 0$ for all $x \in[a, b] \Rightarrow\|f\| \geq 0$
(2) ) Let $f \in X$
$\|f\|=0 \Leftrightarrow \max \{|f(x)|: 0 \leq x \leq 1\}=0 \Leftrightarrow|f(x)|=0 \quad \forall x \in[a, b] \quad \Leftrightarrow f(x)=0 \quad \forall x \in[a, b] \Leftrightarrow f=0$
(3) Let $f \in X$ and $\lambda \in \mathbb{R}$

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$\|\lambda f\|=\max \{|(\lambda f)(x)|: a \leq x \leq b\}=\max \{|\lambda \| f(x)|: a \leq x \leq b\}=|\lambda| \max \{|f(x)|: a \leq x \leq b\}=|\lambda|\|f\|$
(4) Let $f, g \in X$

$$
\begin{aligned}
\|f+g\| & =\max \{|(f+g)(x)|: a \leq x \leq b\}=\max \{|f(x)+g(x)|: a \leq x \leq b\} \\
& \leq \max \{|f(x)|+|g(x)|: a \leq x \leq b\} \\
& \leq \max \{|f(x)|: a \leq x \leq b\}+\max \{|g(x)|: a \leq x \leq b\}=\|f\|+\|g\|
\end{aligned}
$$

$\Rightarrow \quad X$ is a normed space. Second to show $X$ is a Banach space.

## Example(3.2.9)

Let $X=C[0,1]$, we define the function $\|\cdot\|: X \rightarrow \mathbb{R}$ by $\|f\|=\int_{0}^{1}|f(x)| d x$ for all $\}_{\cdot} \in X$. Show that $X$ is normed space, but not Banach
Ans :
First, to show $X$ is a normed space.
(1) Since $|f(x)| \geq 0$ for all $x \in[0,1] \Rightarrow\|f\| \geq 0$
(2) (a) If $f=0$, then $\|f\|=\int_{0}^{1}|f(x)| d x=0 \quad$ (b) If $\|f\|=0$, then $\int_{0}^{1}|f(x)| d x=0$

Since $|f(x)| \geq 0$ and $f$ is continuous, then $|f(x)|=0^{=} \Rightarrow f=0$
(3)Let $f \in X$ and $\lambda \in \mathbb{R}$
$\left.\|\lambda f\|=\int_{0}^{1}|(\lambda f)(x)| d x=\int_{0}^{1}|\lambda f(x)| d x=\int_{0}^{1}|\lambda \| f(x)|\right) d x=|\lambda| \int_{0}^{1}|f(x)| d x=\mid \lambda\|f\|$
(4) Let $f, g \in X$

$$
\begin{aligned}
& \|f+g\|=\int_{0}^{1}|(f+g)(x)| d x=\int_{0}^{1}|f(x)+\dot{g}(x)| d x \leq \int_{0}^{1}(|f(x)|+|g(x)|) d x=\int_{0}^{1}|f(x)| d x+\int_{0}^{1}|g(x)| d x \\
& \|f+g\| \leq\|f\|+\|g\| \\
& \quad \Rightarrow \quad X \text { is a normed space }
\end{aligned}
$$

We now show that $X_{i}$ is not complete. Consider the sequence $\left\{f_{n}\right\}$ in $X$ defined as follows

$$
f_{n}(x)=\left\{\begin{array}{c}
1,0 \leq x \leq \frac{1}{2} \\
-n x+\frac{1}{2} n+1, \quad \frac{1}{2}<x \leq \frac{1}{2}+\frac{1}{n} \\
0, \quad \frac{1}{2}+\frac{1}{n}<x \leq 1
\end{array}\right.
$$

Then $\left\{f_{n}\right\}$ is a Cauchy sequence in $X$, because , if $m>n \geq 3$, then $\left\|f_{m}-f_{n}\right\|=\int_{0}^{1}\left|\left(f_{m}-f_{n}\right)(x)\right| d x=\int_{0}^{1}\left|f_{m}(x)-f_{n}(x)\right| d x$

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$$
\begin{aligned}
& \left\|f_{m}-f_{n}\right\|=\int_{0}^{\frac{1}{2}}\left|f_{m}(x)-f_{n}(x)\right| d x+\int_{\frac{1}{2}}^{1}\left|f_{m}(x)-f_{n}(x)\right| d x=\int_{0}^{\frac{1}{2}}|1-1| d x+\int_{\frac{1}{2}}^{1}\left|f_{m}(x)-f_{n}(x)\right| d x \\
& \left\|f_{m}-f_{n}\right\| \leq \int_{\frac{1}{2}}^{1}\left|f_{m}(x)\right| d x+\int_{\frac{1}{2}}^{1}\left|f_{n}(x)\right| d x=\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}}\left|-m x+\frac{1}{2} m+1\right| d x+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}}\left|-n x+\frac{1}{2} n+1\right| d x
\end{aligned}
$$

Since $-m x+\frac{1}{2} m+1 \geq 0$ when $\frac{1}{2}<x \leq \frac{1}{2}+\frac{1}{m}$

$$
\left\|f_{m}-f_{n}\right\| \leq \frac{1}{2 m}+\frac{1}{2 n} \Rightarrow\left\|f_{m}-f_{n}\right\| \rightarrow 0 \text { as } n, m \rightarrow \infty
$$

$\Rightarrow\left\{f_{n}\right\}$ is a Cauchy sequence. But this sequence is not convergent,
For if there existed a $f \in X$ such that $f_{n} \rightarrow f \Rightarrow$

$$
f(x)= \begin{cases}1, & 0 \leq x \leq \frac{1}{2} \\ 0, & \frac{1}{2}<x \leq 1\end{cases}
$$

This contradiction, because $f$ is not continuous

## Theorem(3.2.10)

Let $M$ be a subspace of Banach space $X$, then $M$ is Banach space iff it is closed in $X$ Proof :

Suppose $M$ is a Banach space $\Rightarrow M$ is complete space
Let $x \in \bar{M}$, there is a sequence $\left\{x_{n}\right\}$ in $M$ such that $x_{n} \rightarrow x$, hence $\left\{x_{n}\right\}$ is a Cauchy sequence in $M$
Since $M$ is complete, there is $y \in M$ such that $x_{n} \rightarrow y$, but the converge is unique
$\Rightarrow y=x \Rightarrow x \in M \Rightarrow \bar{M} \subseteq M$, then $M$ is closed
Conversely. Suppose that $M$ is a closed set in $X$
Let $\left\{x_{n}\right\}$ be a Cauchy sequence in $M$
Since $M \subset X \Longrightarrow\left\{x_{n}\right\}$ is a Cauchy sequence in $X$
Since $X$ is complete space, there is $x \in X$ such that $x_{n} \rightarrow x$
Since $x_{n} \in M \Rightarrow x \in \bar{M}$
Since $M$ is a closed set in $X$, then $\bar{M}=M \Rightarrow x \in M$
$\Rightarrow\left\{x_{n}\right\}$ is converge sequence in $M$, then $M$ is complete space.

## Theorem(3.2.11)

Every finite dimensional normed space is complete.

## Proof :

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Let $X$ be finite dimensional normed space with $\operatorname{dim} X=n>0$ and let $\left\{x_{1}, \cdots, x_{n}\right\}$ be a basis for $X$ Let $\left\{y_{m}\right\}$ be any Cauchy sequence in $X$.

$$
\begin{equation*}
\left\|y_{m}-y_{l}\right\| \rightarrow 0 \quad \text { as } m, l \rightarrow \infty \tag{1}
\end{equation*}
$$

Since $y_{m}, y_{l} \in X \Rightarrow y_{m}=\sum_{i=1}^{n} \lambda_{i m} x_{i}, \quad \lambda_{i n} \in F$ and $y_{l}=\sum_{i=1}^{n} \lambda_{i l} x_{i}, \quad \lambda_{i l} \in F$
$\Rightarrow y_{m}-y_{l}=\sum_{i=1}^{n}\left(\lambda_{i m}-\lambda_{i l}\right) x_{i}$
Since $\left\{x_{1}, \cdots, x_{n}\right\}$ is linear independent, by lemma of linear combination, there is $c>0$ such that $\left\|y_{m}-y_{l}\right\|=\left\|\sum_{i=1}^{n}\left(\lambda_{i m}-\lambda_{i i}\right) x_{i}\right\| \geq c \sum_{i=1}^{n}\left|\lambda_{i m}-\lambda_{i l}\right|$
From (1) and (2), we have $\quad \sum_{i=1}^{n}\left|\lambda_{i n}-\lambda_{i}\right| \rightarrow 0$ as $m, l \rightarrow \infty$ for $i_{-} 11, \ldots, n$
$\left|\lambda_{i m}-\lambda_{i i}\right| \rightarrow 0 \quad$ as $m, l \rightarrow \infty$ for $i=1, \cdots, n$
For $i=1, \cdots, n \Rightarrow\left\{\lambda_{i n}\right\}$ is Cauchy sequence in $F$
Since $F$ is either $\mathbb{R}$ or $\mathbb{C}$ and each $\mathbb{R}, \mathbb{C}$ and are complete. $\Rightarrow \exists \lambda_{i} \in F$ such that $\lambda_{i n} \rightarrow \lambda_{i}$
Put $y=\sum_{i=1}^{n} \lambda_{i} x_{i} \Rightarrow y_{m} \rightarrow y, y \in X \Rightarrow X$ is complete.

## Corollary (3.2.12)

Every finite dimensional subspace $M$ of a normed space $X$, is closed

## Proof:

Since $M$ is a finite dimensional subspace of a normed space $X \Rightarrow M$ is a complete
space $\Rightarrow M$ is closed
Note that, infinite dimensional subspace of Banach space need not be closed.

## Example (3.2.13)

Let $X=C[0,1]$ and let $\mathcal{M}=\left[\left\{f_{0}, f_{1}, \cdots\right)\right]$ where $f_{i}(x)=x^{i}$ so that $M$ is the set of all polynomials. $M$ is an infinite dimensional subspace of $X$ but not closed in $X$.

### 3.3 Continuity

## Definition (3.3.1)

Let $X$ and $Y$ be two normed spaces. A function $f: X \rightarrow Y$ is said to be continuous at $x_{0} \in X$, if for each $\bar{\varepsilon}>0$, there exists a $\delta>0$ such that $x \in X$

$$
\left\|x-x_{0}\right\|<\delta \text { implies }\left\|f(x)-f\left(x_{0}\right)\right\|<\varepsilon
$$

Or equivalently, a function $f: X \rightarrow Y$ is continuous at $x_{0} \in X$, if for every sequence $\left\{x_{n}\right\}$ in $X$ converging to $x_{0}$, the sequence $\left\{f\left(x_{n}\right)\right\}$ in $Y$ converges to $f\left(x_{0}\right) \in Y$, i.e.

$$
x_{n} \rightarrow x_{0} \Rightarrow f\left(x_{n}\right) \rightarrow f\left(x_{0}\right)
$$

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## Theorem (3.3.2)

Let $X$ be a normed space. Then the function $f: X \rightarrow \mathbb{R}, f(x)=\|x\|$ is continuous, the norm $\|$. on $X$ is a continuous function.

## Proof:

Let $x_{0} \in X$ and $\left\{x_{n}\right\}$ in $X$ such that $x_{n} \rightarrow x_{0}$ as $n \rightarrow \infty$
Now

$$
\left|f\left(x_{n}\right)-f\left(x_{0}\right)\right|=\left|\left\|x_{n}\right\|-\left\|x_{0}\right\|\right| \leq\left\|x_{n}-x_{0}\right\|
$$

Since $x_{n} \rightarrow x_{0}$, then $\left\|x_{n}-x_{0}\right\| \rightarrow 0 \quad$ as $n \rightarrow \infty$
$\left|f\left(x_{n}\right)-f\left(x_{0}\right)\right| \rightarrow 0$ as $n \rightarrow \infty$, i.e. $f\left(x_{n}\right) \rightarrow f\left(x_{0}\right)$
Then $f$ is continuous at $x_{0}$ and $x_{0}$ is any point, hence $f$ is continuous

## Theorem (3.3.3)

Let $X$ be a normed space over $F$. Then the functions $f: X \otimes X \rightarrow X, f(x, y)=x+y$ and is $g: F \times X \rightarrow X, g(\lambda, x)=\lambda x$ are continuous, in other words vector addition and scalar multiplication are jointly continuous.

## Proof :

(1) Let $x_{0}, y_{0} \in X$ and $\left\{x_{n}\right\},\left\{y_{n}\right\}$ in $X$ such that $x_{n} \rightarrow x_{0}$ and $y_{n} \rightarrow y_{0}$ as $n \rightarrow \infty$

Now
$\left\|f\left(x_{n}, y_{n}\right)-f\left(x_{0}, y_{0}\right)\right\|=\left\|\left(x_{n}+y_{n}\right)-\left(x_{0}+y_{0}\right)\right\|=\left\|\left(x_{n}-x_{0}\right)+\left(y_{n}-y_{0}\right)\right\| \leq\left\|x_{n}-x_{0}\right\|+\left\|y_{n}-y_{0}\right\|$
Since $\left\|x_{n}-x_{0}\right\| \rightarrow 0$ and $\left\|y_{n}-y_{0}\right\| \rightarrow 0$ as $n \rightarrow \infty$, we have $\left\|f\left(x_{n}, y_{n}\right)-f\left(x_{0}, y_{0}\right)\right\| \rightarrow 0$ as $n \rightarrow \infty$ $f\left(x_{n}, y_{n}\right) \rightarrow f\left(x_{0}, y_{0}\right) \quad$ as $n \rightarrow \infty$
$f$ is continuous at $\left(x_{0}, y_{0}\right)$ and $\left(x_{0}, y_{0}\right)$ is any point in $X \times X$, hence $f$ is continuous
(2) Let $x_{0} \in X, \lambda \in F$ and $\left\{x_{n}\right\}$, in $X,\left\{\lambda_{n}\right\}$ in $F$ such that $x_{n} \rightarrow x_{0}$ and $\lambda_{n} \rightarrow \lambda$ as $n \rightarrow \infty$ Now

$$
\begin{aligned}
\left\|g\left(\lambda_{n}, x_{n}\right)-g\left(\lambda, x_{0}\right)\right\| & =\left\|\lambda_{n} x_{n}-\lambda x_{0}\right\|=\left\|\left(\lambda_{n} x-\lambda_{n} x_{0}\right)+\left(\lambda_{n} x_{0}-\lambda x_{0}\right)\right\| \\
0 & =\left\|\lambda_{n}\left(x_{n}-x_{0}\right)+\left(\lambda_{n}-\lambda\right) x_{0}\right\| \leq\left|\lambda_{n}\left\|x_{n}-x_{0}\right\|+\right| \lambda_{n}-\lambda\left\|x_{0}\right\|
\end{aligned}
$$

Since $\left\|x_{n}-x_{0}\right\| \rightarrow 0$ and $\left|\lambda_{n}-\lambda\right| \rightarrow 0$ as $n \rightarrow \infty$, we have $\left\|g\left(\lambda_{n}, x_{n}\right)-g\left(\lambda, x_{0}\right)\right\| \rightarrow 0$ as $n \rightarrow \infty$ $g\left(\lambda_{n}, x_{n}\right) \rightarrow g\left(\lambda, x_{0}\right) \quad$ as $n \rightarrow \infty$
$g$ is continuous at ( $\lambda, x_{0}$ ) and ( $\lambda, x_{0}$ ) is any point in $F \times X$, hence $g$ is continuous

## Corollary (3.3.4)

Every normed space $X$ is topological linear space

## Example(3.3.5)

Every normed space is locally convex
Ans:
Let $(X,\| \|)$ be a normed space, then $X$ is topological linear space

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Let $\beta=\left\{B_{r}(0): r>0\right\}$, where $B_{r}(0)=\{x \in X:\|x\|<r\}$,
Let $G$ be an open set in $X$, then $G$ is the union of open balls, so $0 \in B_{r}(0) \subset G$ for some $r>0$, then $\beta$ is a local base at 0 in $X$
Since every open ball is convex set, then $B_{r}(0)$ is convex set for all $r>0$, then $\beta$ is a convex local base at 0 in $X$
Hence ( $X,\| \| \cdot \mid$ ) is locally convex space.

## $3.4 L^{p}$-Spaces

Let $\Omega$ be a non empty set and let $L(\Omega)$ denote the set of all real valued functions defined on $\Omega$. Then $L(\Omega)$ is a linear space under the following addition and scalar multiplication
(1) $(f+g)(x)=f(x)+g(x)$ for all $f, g \in L(\Omega)$
(2) $(\lambda f)(x)=\lambda f(x)$ for all $f \in L(\Omega)$ and for all $\lambda \in \mathbb{R}$

## Definition(3.4.1)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $p$ is a real number with $0<p<\infty$, and let $f$ real valued Borel measurable function defined on $\Omega$. We define

$$
\|f\|_{p}=\left(\int_{\Omega} \mid f^{p} d \mu\right)^{\text {min }}
$$

If $0<p<1$, then $\min \left\{\frac{1}{p}, 1\right\}=1 \Rightarrow\|f\|_{p}=\int_{\Omega}\left|f_{z}\right|^{p} d \mu$.
If $1 \leq p<\infty$, then $\min \left\{\frac{1}{p}, 1\right\}=\frac{1}{p} \Rightarrow\|f\|_{p}=\left(\int_{\Omega}|f|^{p} d \mu\right)^{\frac{1}{p}}$
If $p=\infty$. we set $D=\left\{\alpha \in \mathbb{R}^{+}: \mu\{| | \gg \alpha\}=0\right\}$, i.e.
$D=\left\{\alpha \in \mathbb{R}^{+}: \mu\{x \in \Omega:|f(x)| \geqslant \alpha\}=0\right\}=\left\{\alpha \in \mathbb{R}^{+}:|f| \leq \alpha \quad\right.$ ae. $\left.[\mu]\right\}$. Define

$$
\|f\|_{\infty}=\left\{\begin{array}{cc}
\infty, & D=\phi \\
\inf D, & D \neq \phi
\end{array}\right.
$$

Suppose $\|f\|_{\infty}<\infty$.
Since $\left.f^{-1}\left(\| \| f \|_{\infty}, \infty\right]\right)=\bigcup_{n=1}^{\infty} f^{-1}\left(\left(\|f\|_{\infty}+\frac{1}{n}, \infty\right]\right)$ and $\left.\mu\left(f^{-1}\left(\|f\|_{\infty}+\frac{1}{n}, \infty\right]\right)\right)=$, we see $\|f\|_{\infty} \in D$.
The quantity $\|f\|_{\infty}$ is called the essential supremum of $|f|$. The essential supremum of $f$ is denoted by ess sup $f$ and defined as:
${ }^{2}$ ess sup $f=\inf \left\{\lambda \in R^{*}: f \leq \lambda \quad\right.$ a.e. $\}=\inf \left\{\lambda \in R^{*}: \mu\{x \in \Omega: f(x)>\lambda\}=0\right\},\|f\|_{\infty}=e s s \sup |f|$.
For example , if the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=\left\{\begin{array}{ll}1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}\end{array}\right.$, then

$$
\sup |f|=1, \text { but ess sup }|f|=0
$$

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## Definition(2.4.1)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p$ is a real number with $0<p \leq \infty$. We define the space $L^{p}=L^{p}(\Omega, \mathcal{F}, \mu)$ as the family of all real valued Borel measurable functions $f$ such that $\|f\|_{p}<\infty$. i.e.
(1) If $0<p<1$, then $L^{p}=L^{p}(\Omega, \mathcal{F}, \mu)$ as the family of all real valued Borel measurable-functions $f$ such that $\|f\|_{p}=\int_{\Omega}|f|^{p} d \mu<\infty$
(2) If $1 \leq p<\infty$, then $L^{p}=L^{p}(\Omega, \mathcal{F}, \mu)$ as the family of all real valued Borel measurable functions $f$ such that $\|f\|_{p}=\left(\int_{\Omega}|f|^{p} d \mu\right)^{\frac{1}{p}}<\infty$
(3) If $p=\infty$, then $L^{\infty}=L^{\infty}(\Omega, \mathcal{F}, \mu)$ as the family of all real valued Borel measurable functions $f$ such that $\|f\|_{\infty}=\inf \left\{\alpha \in \mathbb{R}^{+}: \mu\{|f|>\alpha\}=0\right\}<\infty$

## Definition(3.4.3)

Let $P$ and $q$ be positive real numbers, we say that $P$ and $q$ is a pair of conjugate exponents if $\frac{1}{p}+\frac{1}{q}=1$.
It is clear that $\frac{1}{p}+\frac{1}{q}=1$ implies
(1) $1<p<\infty$ and $1<q<\infty$
(2) If $p=2$, then $q=2$
(3) As $p \rightarrow 1$, then $q \rightarrow \infty$

Theorem(3.4.4)
If $a \geq 0, b \geq 0, p>1, q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then $a b \leq \frac{1}{p} a^{p}+\frac{1}{q} b^{q}$ and $a b=\frac{1}{p} a^{p}+\frac{1}{q} b^{q}$ iff $a^{p}=b^{q}$

## Proof :

Let $f(t)=1-\lambda+\lambda t-t^{\lambda}$ such that $t \geq 0, \lambda=\frac{1}{p}$
$f^{\prime}(t)<0$ for all $0<t<1$ and $f^{\prime}(t)>0$ for all $\Rightarrow t>0 \Rightarrow f(t) \geq f(1)=0$

$$
\Rightarrow 1-\lambda+\lambda t-t^{\lambda} \geq 0 \Rightarrow t^{\lambda} \leq 1-\lambda+\lambda t
$$

If $b=0$, then $a b=0 \leq \frac{1}{p} a^{p}$ and if $b>0$, put $t=\frac{a^{p}}{b^{q}} \Rightarrow\left(\frac{a^{p}}{a^{q}}\right)^{\lambda} \leq 1-\lambda+\lambda \frac{a^{p}}{b^{q}}$
Since $\lambda=\frac{1}{p} \Rightarrow\left(\frac{a^{p}}{a^{q}}\right)^{\frac{1}{p}} \leq 1-\frac{1}{p}+\frac{1}{p} \frac{a^{p}}{b^{q}} \Rightarrow b^{q}\left(\frac{a^{p}}{a^{q}}\right)^{\frac{1}{p}} \leq\left(1-\frac{1}{p}\right) b^{q}+\frac{1}{p} a^{p}$
Since $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow 1-\frac{1}{p}=\frac{1}{q}$ and $b^{q}\left(\frac{a^{p}}{a^{q}}\right)^{\frac{1}{p}}=a b^{q-\frac{q}{p}}=a b^{\frac{1}{q}}=a b \Rightarrow a b \leq \frac{1}{p} a^{p}+\frac{1}{q} b^{q}$

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## Theorem (3.4.5) Holder's inequality

Let $1<p<\infty$ and let $q$ be its conjugate exponent. If $f \in L^{p}(\Omega, \mathcal{F}, \mu)$ and $g \in L^{q}(\Omega, \mathcal{F}, \mu)$, then $f g \in L^{1}(\Omega, \mathcal{F}, \mu)$ and $\|f g\|_{1} \leq\|f\|_{p}\|g\|_{q}$

## Proof :

If $\|f\|_{p}=0$ or $\|g\|_{q}=0$, the inequality is immediate
If $\|f\|_{p} \neq 0$ and $\|g\|_{q} \neq 0$, take $a=\frac{|f(x)|}{\|f\|_{p}}, \quad b=\frac{|g(x)|}{\|g\|_{q}}$
Since $a b \leq \frac{1}{p} a^{p}+\frac{1}{q} b^{q}$, we have

$$
\begin{aligned}
& \frac{|f(x)|}{\|f\|_{p}} \times \frac{|g(x)|}{\|g\|_{q}} \leq \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_{p}}\right)^{p}+\frac{1}{q}\left(\frac{|g(x)|_{q}^{q}}{\|g\|_{q}} \Rightarrow \frac{|f(x) g(x)|}{\|f\|_{p}\|g\|_{q}} \leq \frac{|f(x)|^{p}}{p\|f\|_{o}^{p}}+\frac{|g(x)|^{0}}{q\|g\|_{q}^{q}}\right. \\
& \quad \Rightarrow \int_{\Omega} \frac{|f(x) g(x)|}{\|f\|_{p}\|g\|_{q}} d \mu \leq \int_{\Omega}\left(\frac{|f(x)|^{p}}{p\|f\|_{p}^{p}}+\frac{|g(x)|^{q}}{q\|g\|_{q}^{q}}\right) d \mu=\int_{\Omega} \frac{|f(x)|^{p}}{p\| \|_{p}^{p}} d \mu+\int_{\Omega} \frac{|g(x)|^{q}}{q\|g\|_{q}^{q}} d \mu \\
& \Rightarrow \frac{1}{\|f\|_{p}\|g\|_{q}} \int_{\Omega}|f(x) g(x)| d \mu \leq \frac{\|f\|_{p}^{p}}{p\|f\|_{p}^{p}}+\frac{\|g\|_{q}^{q}}{q\|g\|_{q}^{q}}=\frac{1}{p}+\frac{1}{\sigma^{q}}=1
\end{aligned}
$$

$$
\Rightarrow \frac{1}{\|f\|_{p}\|g\|_{q}}\|f\|_{1} \leq 1 \quad \Rightarrow\|f g\|_{1} \leq\|f\|_{p}\|g\|_{q}
$$

Corollary (3.4.6) Cauchy-Schwarž Inequality
. If $f, g \in L^{2}(\Omega, \mathcal{F}, \mu)$, then $f g \in L^{1}(\Omega, \mathcal{F}, \mu)$ and $\|f g\|_{1} \leq\|f\|_{2}\|g\|_{2}$

## Proof:

Take $p=2 \Rightarrow q=2$.
By using Holder's inequality, we have $f g \in L^{\prime}(\Omega, \mathcal{F}, \mu)$ and $\|f g\|_{1} \leq\|f\|_{2}\|g\|_{2}$

## Theorem (3.4.7)

Let $a \geq 0, b \geq 0$
(1) If $0<p \leq 1$, then $(a+b)^{p} \leq a^{p}+b^{p}$
(2) If $p \geq 1$, then $(a+b)^{p} \leq 2^{p-1}\left(a^{p}+b^{p}\right)$

## Proof :

(1) Since $\left(\frac{a}{a+b}\right)^{p}+\left(\frac{b}{a+b}\right)^{p} \geq \frac{a}{a+b}+\frac{b}{a+b}=\frac{a+b}{a+b}=1$.
$\Rightarrow \frac{a^{p}}{(a+b)^{p}}+\frac{b^{p}}{(a+b)^{p}} \geq 1 \Rightarrow \frac{a^{p}+b^{p}}{(a+b)^{p}} \geq 1 \Rightarrow(a+b)^{p} \leq a^{p}+b^{p}$
(2) Let $f(x)=\frac{\lambda\left((a+x)^{p}-2^{p-1}\left(a^{p}+x^{p}\right)\right)}{\lambda x} \Rightarrow f(x)=p(a+x)^{p-1}-2^{p-1} p x^{p-1}$

Since $p \geq 1$,

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$f(x)>0$ for $x<a, f(x)=0$ at $x=a, f(x)<0$ for $x>a$
The maximum therefore occurs at $x=a$, hence $(a+b)^{p}-2^{p-1}\left(a^{p}+b^{p}\right) \leq(a+a)^{p}-2^{p-1}\left(a^{p}+a^{p}\right)=0$
Theorem (3.4.8) Minkokowsk's Inequality
Let $1 \leq p<\infty$ and let $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$, then $f+g \in L^{p}(\Omega, \mathcal{F}, \mu)$ and $\|f+g\|_{p} \leq\|f\|_{p}+\|g\|_{q}$

## Proof :

$$
\text { Since }|f+g| \leq|f|+|g| \Rightarrow|f+g|^{p} \leq(|f|+|g|)^{p} \leq 2^{p-1}\left(|f|^{p}+|g|^{p}\right)
$$

Since $f, g \in L^{p}(\Omega, \mathcal{F}, \mu) \Rightarrow f+g \in L^{p}(\Omega, \mathcal{F}, \mu)$.
Now the inequality is clear when $p=1$, so assume $p>1$ and choose $q$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then $|f+g|^{p}=|f+g||f+g|^{p-1} \leq(|f|+|g|)|f+g|^{p-1}=|f||f+g|^{p-1}+|g||f+g|^{p-1}$
Now $|f+g|^{p-1} \in L^{q}(\Omega, \mathcal{F}, \mu)$, for $(p-1) q=\frac{p-1}{\frac{1}{q}}=\frac{p-1}{1-\frac{1}{p}}=\frac{p-1}{\frac{p-1}{p}}=p$,
hence $\int_{\Omega}\left(|f+g|^{p-1}\right)^{q} d \mu=\int_{\Omega}\left(|f+g|^{p} d \mu<\infty\right.$
Since $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$ and $|f+g|^{p-1} \in L^{q}(\Omega, \mathcal{F}, \mu)$. By using Holder's inequality, we have $|f||f+g|^{p-1},|g \| f+g|^{p-1} \in L^{1}(\Omega, \mathcal{F}, \mu)$ and
$\int_{\Omega}|f||f+g|^{p-1} d \mu \leq\|f\|_{p}\left(\int_{\Omega}\left(|f+g|^{p-1}\right)^{q} d \mu\right)^{\frac{1}{q}}=\|f\|_{p}\|f+g\|_{p}^{\frac{p}{q}}$ and $\int_{\Omega}\left|g\left\|f+\left.g\right|^{p-1} d \mu \leq\right\| g\left\|_{p}\right\| f+g \|_{p}^{\frac{p}{q}}\right.$
Since $|f+g|^{p} \leq|f||f+g|^{p-1}+|g \| f+g|_{i}^{p-1}$, we have $\|f+g\|_{p}^{p} \leq\left(\|f\|_{p}+\|g\|_{p}\right)\left(\|f+g\|_{p}^{p}\right)$
Since $p-\frac{p}{q}=1$, the result follows.

## Corollary (3.4.9)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p$ is a real number with $0<p \leq \infty$. Then $L^{p}(\Omega, \mathcal{F}, \mu)$ is a subspace of $L(\Omega)$

## Proof :

Let $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbb{R}$, by using Minkokowsk's Inequality, we have

$$
\alpha f+\beta g \in L^{p}(\Omega, \mathcal{F}, \mu)
$$

## Theorem(3.4.10)

is a seminorm on $L^{p}(\Omega, \mathcal{F}, \mu)$ where $1 \leq p \leq \infty$. i.e.
(1) $\|f\|_{p} \geq 0$ for all $f \in L^{p}(\Omega, \mathcal{F}, \mu)$
(2) If $f=0$, then $\|f\|=0$
(3) $\|\lambda f\|=|\lambda|\|f\|$ for all $f \in L^{p}(\Omega, \mathcal{F}, \mu)$ and all $\lambda \in \mathbb{R}$
(4) $\|f+g\|_{p}=\|f\|_{p}+\|g\|_{p}$ for all $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$

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## Proof :

The first two properties follow from the definition of $\left\|\|_{p}\right.$, and the last property is consequence of Minkokowsk's Inequality.

## Remark

We can change $\left\|\|_{p}\right.$ into a norm by passing to equivalence as follows.
If $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$, define $f \sim g$ iff $f=g$ a.e. $\Rightarrow\|f\|_{p}=\|g\|_{p}$. Then $\|f\|_{p}$ is the šame for all $f$ in given equivalence class. Thus $L^{p}(\Omega, \mathcal{F}, \mu)$ is the family of equivatence class, $L^{p}(\Omega, \mathcal{F}, \mu)$ becomes a linear space, and $\left\|\|_{p}\right.$ is a seminorm on $L^{p}(\Omega, \mathcal{F}, \mu)$. In fact $\| \|_{p}$ is a norm, since $\|f\|_{p}=0$ implies $f=0$ a.e.
Since every normed space is a metric space, so $L^{p}(\Omega, \mathcal{F}, \mu)$ is a metric space.

## Remark

If $0<p<1,\| \|_{p}$ is not a seminorm on $L^{p}(\Omega, \mathcal{F}, \mu)$.

## Definition(3.4.11)

A sequence $\left\{f_{n}\right\}$ in $L^{p}(\Omega, \mathcal{F}, \mu)$ is said to be
(1) convergent in $L^{p}(\Omega, \mathcal{F}, \mu)$ if there is $f \in L^{p}(\Omega, \mathcal{F}, \mu)$ such that $\left\|f_{n}-f\right\|_{p} \rightarrow 0$ as $n \rightarrow \infty$.
(2) Cauchy in $L^{p}(\Omega, \mathcal{F}, \mu)$ if $\left\|f_{n}-f_{m}\right\|_{p} \rightarrow 0$ as $n, m \rightarrow \infty$.

## Theorem(3.4.12)

Let $\left\{f_{n}\right\}$ be a sequence in $L^{p}(\Omega, \mathcal{F}, \mu), 0<p<\infty$ satisfying $\left\|f_{n}-f_{n+1}\right\|_{p}<\left(\frac{1}{4}\right)^{n}, n=1,2, \cdots$.
Then $\left\{f_{n}\right\}$ converges a.e.

## Proof :

Set $A_{n}=\left\{x \in \Omega:\left|f_{n}(x)-f_{n+1}(x)\right|>2^{-n}\right\}$. By Chebyshev's inequality, we have $\mu\left(A_{n}\right) \leq 2^{n_{p}} \int_{\Omega}\left|f_{n}-f_{n^{p}+1}\right|^{p} d \bar{\mu} \leq\left(\frac{1}{4}\right)^{n_{p}} \times 2^{n_{p}}=2^{-n_{p}} \quad$. This show that $\sum_{n=1}^{\infty} \mu\left(A_{n}\right)<\infty$
By Borel Cantelli lemma, we have $\mu\left(\lim _{n \rightarrow \infty} \sup _{n}\right)=0$
But if $x \notin \lim \sup _{n} A_{n}$, then $\left|f_{n}(x)-f_{n+1}(x)\right|<2^{-n}$ for large $n$, so $\left\{f_{n}(x)\right\}$ is Cauchy sequence in $\mathbb{R}$ and hence $\left\{f_{n}(x)\right\}$ converges .

## Theorem(3.4.13)

$L^{p}(\Omega, \mathcal{F}, \mu)$, where $1 \leq p<\infty$ is a Banach space

## Proof :

Since $L^{p}(\Omega, \mathcal{F}, \mu)$ is a normed space. We need to show that $L^{p}(\Omega, \mathcal{F}, \mu)$ is complete Let $\left\{f_{n}\right\}$ be a Cauchy sequence in $L^{p}(\Omega, \mathcal{F}, \mu)$, where $1 \leq p<\infty$

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Given any $\varepsilon>0$, there is $N \in \mathbb{Z}^{+}$such that $\left\|f_{n}-f_{m}\right\|_{p}<\varepsilon$ for all $n, m>N$
The for each $k=1,2, \cdots$ there is a $n_{k}$ such that $\left\|f_{n}-f_{m}\right\|_{p}<\left(\frac{1}{4}\right)^{n}$ for all $n, m>n_{k}$
By theorem(3.4.12), we have $f_{n_{k}}(x) \rightarrow f$ a.e. . We need to show that $f \in L^{p}(\Omega, \mathcal{F}, \mu)$ and that it is the $L^{p}(\Omega, \mathcal{F}, \mu)$ limit of $\left\{f_{n}\right\}$. Let $\varepsilon>0$. Take $N$ so large that $\left\|f_{n}-g_{m}\right\|_{p}<\varepsilon$ for all $\left.n, m\right\}_{N}$
Fix such an $m$. Then by the pointwise convergence of the subsequence and by Fatou's lemma we have

$$
\int_{\Omega}\left|f-f_{m}\right|^{p} d \mu=\int_{\Omega} \lim _{k \rightarrow \infty}\left|f_{n_{k}}-f_{m}\right|^{p} d \mu \leq \liminf _{k \rightarrow \infty} \int_{\Omega}\left|f_{n_{k}}-f_{m}\right|^{p} d \mu<\varepsilon^{p}
$$

Therefore $f_{n} \rightarrow f$ in $L^{p}(\Omega, \mathcal{F}, \mu)$ and $\int_{\Omega} \mid f-f_{m} \phi \mu<\infty$ for $m$ sufficiently large. But then, $\|f\|_{p}=\left\|f_{m}-f_{m}-f\right\|_{p} \leq\left\|f_{m}\right\|_{p}+\left\|f_{m}-f\right\|_{p}$, which shows that $f \in L^{p}(\Omega, \mathcal{F}, \mathfrak{M})$.

## Remark

Let $\Omega$ is an arbitrary set, $\mathcal{F}$ consists of all subsets of $\Omega, \mu$ is counting measure,(i.e. if $A$ has $n$ numbers, $n=0,1,2, \cdots$, then $\mu(A)=n$; if $A$ is an infinite set, then $\mu(A)=\infty)$.
If $f=(f(\alpha): \alpha \in \Omega)$ is a nonnegative real-valued function on $\Omega$, then

$$
\begin{equation*}
\int_{\Omega} f d \mu=\sum_{\alpha} f(\alpha) \tag{1}
\end{equation*}
$$

where the series is defined as $\sup \left\{\sum_{\alpha} f(\alpha): A \subset \Omega, A\right.$ is finite $\}$. If $f(\alpha)>0$ for uncountably many $\alpha$, then for some $\delta>0$ we have $f(\alpha) \geq \delta$ for infinitely many $\alpha$, so that $\sum_{\alpha} f(\alpha)=\infty$.

- If the nonnegative hypothesis is dropped, we apply the above results to $f^{+}$and $f^{-}$to again
obtain equation (1), where the series is interpreted as

$$
\sum_{\alpha} f^{+}(\alpha)-\sum_{\alpha} f^{-}(\alpha)
$$

- The space $L^{p}(\Omega, \mathcal{F}, \mu)$ will be dented by $\ell^{p}(\Omega)$, it consists of all real-valued functions $f=(f(\alpha): \alpha \in \Omega)$ such that $f(\alpha)=0$ for all but countably many $\alpha$ and

$$
\|f\|_{p}=\left(\sum_{n=1}^{\infty}|f(\alpha)|^{p}\right)^{\frac{1}{p}}<\infty
$$

- If $\Omega$ is the set of positive integers, the space $\ell^{p}(\Omega)$ will be dented simply by $\ell^{p}$, it consists of all sequences $f=\left\{x_{n}\right\}$ such that

$$
\|f\|_{p}=\left(\sum_{n=1}^{\infty}\left|x_{n}\right|^{p}\right)^{\frac{1}{p}}<\infty
$$

It will be useful to state the Holder and Minkowski inequalities for sums

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- If $f \in L^{p}(\Omega)$ and $g \in L^{q}(\Omega)$ where $p$ and $q$ are conjugate exponent, $1 \leq p<\infty$, then $f g \in L^{1}(\Omega)$ and $\quad \sum_{\alpha}|f(\alpha) g(\alpha)| \leq\left(\sum_{\alpha}|f(\alpha)|^{p}\right)^{\frac{1}{p}}\left(\sum_{\alpha}|g(\alpha)|^{q}\right)^{\frac{1}{q}}$
- If $f, g \in L^{p}(\Omega), 1 \leq p<\infty$, then $f+g \in L^{p}(\Omega)$ and

$$
\left(\sum_{\alpha}|f(\alpha)+g(\alpha)|^{p} \leq\left(\sum_{\alpha}|f(\alpha)|^{p}\right)^{\frac{1}{p}}+\left(\sum_{\alpha}|g(\alpha)|^{p}\right)^{\frac{1}{p}}\right.
$$

## Theorem(3.4.14)

(1) A sequence $\left\{f_{n}\right\}$ in $L^{\infty}(\Omega, \mathcal{F}, \mu)$ is converges to $f \in L^{\infty}(\Omega, \mathcal{F}, \mu)$ iff there is a set $A \in \mathcal{F}$ with $\mu(A)=0$ such that $f_{n} \xrightarrow{u} f$ on $A^{c}$.
(2) A sequence $\left\{f_{n}\right\}$ is Cauchy in $L^{\infty}(\Omega, \mathcal{F}, \mu)$ iff there is a set $A \in \mathcal{F}$ with $\mu(A)=0$ such that $\left\{f_{n}\right\}$ is uniformly Cauchy on $A^{c}$.

## Proof :

Suppose that $f_{n} \rightarrow f$ in $L^{\infty}(\Omega, \mathcal{F}, \mu) \Rightarrow\left\|f_{n}-f\right\|_{\infty} \rightarrow 0$
Then for each $k>1$ there is an $n>n_{k}$ sufficiently large so that $\left\|f_{n}-f\right\|_{\infty}<\frac{1}{k}$. Thus, there is a set $A_{k}$ so such that $\mu\left(A_{k}\right)=0$ and $\left|f_{n}(x)-f(x)\right|<\frac{1}{k}$ for every $x \in A_{k}^{c}$. Let $A=\bigcup_{k=1} A_{k}$. Then $\mu(A)=0$ and $f_{n} \rightarrow f$ uniformly on $A^{\circ} \mathrm{C}$.
Conversely: Suppose $f_{n} \rightarrow f$ uniformly on $A^{c} f_{n} \rightarrow f$ uniformly on $A^{c}$ and $\mu(A)=0$.
Then given $\varepsilon>0$ there is an $N$ such that for all $n>N$ and $x \in A^{c},\left|f_{n}(x)-f(x)\right|<\varepsilon$. This is the same as saying that $\left\|f_{n}-f\right\|_{\infty}<\dot{\varepsilon}$ for all $n>N$.

## Theorem(3.4.15)

$L^{\infty}(\Omega, \mathcal{F}, \mu)$ is a Banach space

## Proof:

Since $L^{\infty}(\Omega, \mathcal{F}, \mu)$ is a normed space. We need to show that $L^{\infty}(\Omega, \mathcal{F}, \mu)$ is complete
Let $\left\{f_{n}\right\}$ be a Cabehy sequence in $L^{\infty}(\Omega, \mathcal{F}, \mu)$, by theorem(3.4.14), there is a set $A \in \mathcal{F}$ with $\mu(A)_{s}=0$ such that $\left\{f_{n}\right\}$ is uniformly Cauchy on $A^{c}$. That is, given any $\varepsilon>0$, there is $N \in \mathbb{Z}^{+}$such that for all $n, m>N$ and all $x \in A^{c},\left|f_{n}(x)-f_{m}(x)\right|<\varepsilon$.
Therefore the sequence $\left\{f_{n}\right\}$ converges uniformly on $A^{c}$ to a function $f$. Define $f(x)=0$ for $x \in A$. Then $\left\{f_{n}\right\}$ converges to $f$ in $L^{\infty}(\Omega, \mathcal{F}, \mu)$ and $f \in L^{\infty}(\Omega, \mathcal{F}, \mu)$

## Remark

If $\Omega$ is an arbitrary set, $\mathcal{F}$ consists of all subsets of $\Omega, \mu$ is counting measure, then $L^{\infty}(\Omega, \mathcal{F}, \mu)$ is the set of all bounded real-valued functions $f=(f(\alpha): \alpha \in \Omega)$, dented by $\ell^{\infty}(\Omega)$. The essential

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supremum is simply the supremum; in other words, $\|f\|_{\infty}$ sup $\{f(\alpha): \alpha \in \Omega \mid\}$ is the set of positive integers, $\ell^{\infty}(\Omega)$ is the space of bounded sequences of real numbers denoted by $\ell^{\infty}$

## Theorem(3.4.16)

If $0<p<1$. Define $d_{p}: L^{p}(\Omega, \mathcal{F}, \mu) \times L^{p}(\Omega, \mathcal{F}, \mu) \rightarrow \mathbb{R}$ by

$$
d_{p}(f, g)=\int_{\Omega}|f-g|^{p} d \mu \text { for all } f, g \in L^{p}(\Omega, \mathcal{F}, \mu)
$$

Then $\left(L^{p}(\Omega, \mathcal{F}, \mu), d_{p}\right)$ is a metric space, i.e.
(1) $d_{p}(f, g) \geq 0$ for all $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$
(2) $d_{p}(f, g)=d_{p}(g, f)$ for all $f, g \in L^{p}(\Omega, \mathcal{F}, \mu)$
(3) $d_{p}(f, g)=0$ if $f=g$
(4) $d_{p}(f, g)=0$ implies only that $f=g$ a.e.
(5) $d_{p}(f, h) \leq d_{p}(f, g)+d_{p}(g, h)$ for all $f, g, h \in L^{p}(\Omega, \mathcal{F}, \mu)$

## Theorem(3.4.17)

If $0<p<1$. Then $L^{p}(\Omega, \mathcal{F}, \mu)$ is complete metric space

## Example(3.4.18)

The space $L^{p}$ with $0<p<1$ is not locally conyex
Ans :
$L^{p}=\{f:[0,1] \rightarrow \mathbb{R}$, where $f$ is measurable function $\}$ for which

$$
\Delta(f)=\int_{0}^{1}|f(x)|^{p} d x<\infty
$$

Since $0<p<1$, the inequality $(a+b)^{p} \leq a^{p}+b^{p}$ holds when $a \geq 0, b \geq 0$. This gives
$\Delta(f+g) \leq \Delta(f)+\Delta(g)$.
Define $d: L^{p} \times L^{p} \rightarrow R$ by $d(f, g)=\Delta(f-g)$ for all $f, g \in L^{p}$. Then $d$ is an invariant on $L^{p}$. The balls

$$
\beta_{r}=\left\{f \in L^{p}: \Delta(f)<r\right\}
$$

Form a local base for the topology of $L^{p}$,
We claim that $L^{p}$ contains no convex open sets, other than $\phi$ and $L^{p}$.
To prove this, suppose $V \neq \phi$ is open and convex in $L^{p}$. Assume $0 \in V$, without loss of generality. Then $\beta_{r} \subset V$ for some $r>0$.
Pick $f \in L^{P}$. Since $P<1$, there is a positive integer $n$ such that $n^{p-1} \Delta(f)<r$.
By the continuity of the indefinite integral of $|f|^{p}$, there are points $0=x_{\circ}<x_{1}<\cdots<x_{n}=1$ such that

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$$
\int_{x_{k-1}}^{x_{k}}|f(x)|^{p} d x=n^{-1} \Delta(f) \quad, \quad k=1,2, \cdots, n
$$

Define $g_{k}(x)=\left\{\begin{array}{cc}n f(x) & x_{k-1}<x<x_{n} \\ 0, & \text { o.w }\end{array}\right.$
So that $g_{k} \in V$ because $\Delta\left(g_{k}\right)=n^{p-1} \Delta(f)<r, k=1,2, \cdots, n$, we have $\beta_{\gamma} \subset V$
Since $V$ is convex and $f=\frac{1}{n}\left(g_{1}+\cdots+g_{n}\right)$
it follows that $f \in V \Rightarrow L^{P} \subset V$ Hence $L^{P}=V$.

### 3.5 Hilbert Spaces

## Definition(3.7.1)

Let $X$ be linear space over the filed $F$. An inner product on $X$ is a function $\langle\rangle:, X \times X \rightarrow F$ such that for all $x, y, z \in X$ and $\alpha, \beta \in F$ the following are satisfied
(1) $\langle x, x\rangle \geq 0$
(2) $\langle x, x\rangle=0$ iff $x=0$
(3) $\overline{\langle x, y\rangle}=\langle y, x\rangle$ where $\overline{\langle x, y\rangle}$ denote the conjugate complex of the number $\langle x, y\rangle$
(4) $\langle\alpha x+\beta y, z\rangle=\alpha\langle x, z\rangle+\beta\langle y, z\rangle$

The linear space $X$ over the filed $F$ together with $\langle$,$\rangle is called a pre-Hilbert (or inner$ product) space and is denoted by ( $\dot{\bar{x}},\langle$,$\rangle ) or simply X$.

## Remark

Every subspace of pre-Hilbert space is also pre-Hilbert space

## Example (3.5.2)

(1) Let $X=F^{n}$. Dêfine $\langle\rangle:, X \times X \rightarrow F$ by $\langle x, y\rangle=\sum_{i=1}^{n} x_{i} \overline{y_{i}}$ for all $x=\left(x_{1}, x_{2}, \cdots, x_{n}\right) \in X$, $y=\left(y_{1}, y_{2}, \cdots, y_{n}\right) \in X$. Then $X$ is a pre-Hilbert space
(2) Let $X \xlongequal{=\ell^{2}}=\left\{x=\left(x_{1}, \cdots, x_{n}, \cdots\right): x_{i} \in F \quad \forall i, \sum_{i=1}^{\infty}\left|x_{i}\right|<\infty\right\}$. Define $\langle\rangle:, X \times X \rightarrow F \quad$ by $\langle x, y\rangle=\sum_{i=1}^{\infty} x_{i} \overline{y_{i}}$ for all $x=\left(x_{1}, x_{2}, \cdots\right), y=\left(y_{1}, y_{2}, \cdots\right) \in X$. Then $X$ is a pre-Hilbert space (3) Let $X=C[a, b]$. Define $\langle\rangle:, X \times X \rightarrow \mathbb{R}$ by $\langle f, g\rangle=\int_{a}^{b} f(x) \overline{g(x)} d x$ for all $f, g \in X$. Then $X$ is a pre-Hilbert space.
Ans:

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(1) Let $x, y, z \in X$ and $\alpha, \beta \in F$
(a) $\langle x, x\rangle=\sum_{i=1}^{n} x_{i} \bar{x}_{i}=\sum_{i=1}^{n} x_{i}^{2} \geq 0$
(b) $\langle x, x\rangle=0 \Leftrightarrow \sum_{i=1}^{n} x_{i}^{2}=0 \Leftrightarrow x_{i}=0$ for all $i=1,2, \cdots, n \Leftrightarrow x=0$
(c) $\overline{\langle x, y\rangle}=\overline{\sum_{i=1}^{n} x_{i} \bar{y}_{i}}=\sum_{i=}^{n} x_{i} y_{i}=\sum_{i=}^{n} y_{i} \bar{x}_{i}=\langle y, x\rangle$
(d) $\langle\alpha x+\beta y, z\rangle=\sum_{i=1}^{n}(\alpha x+\beta y) \bar{z}=\alpha \sum_{i=1}^{n} x \bar{z}+\beta \sum_{i=1}^{n} y \bar{z}=\alpha\langle x, z\rangle+\beta\langle y, z\rangle$

Then $X$ is a pre-Hilbert space.

## Theorem(3.5.3)

Let $X$ be pre-Hilbert space over the filed $F$
(1) $\langle 0, x\rangle=\langle x, 0\rangle=0$ for all $x \in X$
(2) $\langle x, \alpha y+\beta z\rangle=\bar{\alpha}\langle x, y\rangle+\bar{\beta}\langle x, z\rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathcal{F}$
(3) If $\langle x, z\rangle=\langle y, z\rangle$ for all $z \in X$, then $x=y$

Proof:
(1) $\langle 0, x\rangle=\langle 0 \times 0, x\rangle=0\langle 0, x\rangle=0$
(2)
$\langle x, \alpha y+\beta z\rangle=\overline{\langle\alpha y+\beta z, x\rangle}=\overline{\alpha\langle y, x\rangle+\beta\langle z, x\rangle}=\bar{\alpha}\langle y, x\rangle+\bar{\beta}\langle z, x\rangle=\bar{\alpha} \overline{\langle y, x\rangle}+\bar{\beta}\langle\overline{\langle z, x\rangle}=\bar{\alpha}\langle x, y\rangle+\bar{\beta}\langle x, z\rangle$
(3) $\langle x, z\rangle=\langle y, z\rangle$ for all $z \in X \Rightarrow\rangle\langle x, z\rangle-\langle y, z\rangle=0 \Rightarrow\langle x-y, z\rangle=0$ for all $z \in X$

In particular, put $z=x-y \Rightarrow x-y \in X \Rightarrow\langle x-y, x-y\rangle=0 \Rightarrow x-y=0 \Rightarrow x=y$

## Corollary (3.5.4)

Let $X$ be pre-Hilbert space over the filed $F$
(1) $\left\langle\sum_{i=1}^{n} \alpha_{i} x_{i}, y\right\rangle=\sum_{i=1}^{n} \alpha_{i}\left\langle x x_{i}, y\right\rangle$
(2) $\left\langle x, \sum_{j=1}^{m} \beta_{j} y_{j}\right\rangle=\sum_{j=1}^{m} \overline{\beta_{j}}\left\langle x, y_{j}\right\rangle$
(3) $\left\langle\sum_{i=1}^{n} \alpha_{i} x_{i}, \sum_{j=1}^{m} \beta_{j} y_{j}\right\rangle=\sum_{i, j} \alpha_{i} \overline{\beta_{j}}\left\langle x_{i}, y_{j}\right\rangle$

## Theorem(3.5.5) Cauchy-Schwarz inequality

Let $X$ be pre-Hilbert space over the filed $F$. Define $\|\cdot\|: X \rightarrow \mathbb{R}$ by $\|x\|=\sqrt{\langle x, x\rangle}$ for all $x \in X$, then $|\langle x, y\rangle| \leq\|x\|\|y\|$ for all $x, y \in X$.

## Proof :

If $x=0$ or $y=0$, then $\langle x, y\rangle=0$ and the conclusion is clear.

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Suppose $y \neq 0$. Let $z=\frac{y}{\|y\|} \Rightarrow\|z\|=1$, we shall prove $|\langle x, z\rangle| \leq\|x\|$
For each $\lambda \in F$, then $\langle x-\lambda z, x-\lambda z\rangle \geq 0$
Since $\langle x-\lambda z, x-\lambda z\rangle=\|x\|^{2}-\bar{\lambda}\langle x, z\rangle-\lambda\langle z, x\rangle+|\lambda|^{2}\|z\|$, then $\|x\|^{2}-\bar{\lambda}\langle x, z\rangle-\lambda\langle z, x\rangle+|\lambda|^{2} \geq 0$
$\Rightarrow\|x\|^{2}-\langle x, z\rangle \overline{\langle x, z\rangle}+\langle x, z\rangle \overline{\langle x, z\rangle}-\bar{\lambda}\langle x, z\rangle-\lambda\langle z, x\rangle+|\lambda|^{2} \geq 0$ For all $\lambda \in F$
$\Rightarrow\|x\|^{2}-|\langle x, z\rangle|^{2}+\langle x, z\rangle(\overline{(x, z\rangle}-\bar{\lambda})-\lambda(\langle z, x\rangle-\bar{\lambda}) \geq 0$ For all $\lambda \in F$
$\Rightarrow\|x\|^{2}-|\langle x, z\rangle|^{2}+\langle x, z\rangle(\overline{(\langle x, z\rangle-\lambda})-\lambda(\overline{\langle x, z\rangle-\lambda}) \geq 0$ For all $\lambda \in F$
$\Rightarrow\|x\|^{2}-\mid\langle x, z\rangle^{2}+(\langle x, z\rangle-\lambda)(\overline{(x, z\rangle-\lambda)} \geq 0$ For all $\lambda \in F$
$\Rightarrow\|x\|^{2}-|\langle x, z\rangle|^{2}+|\langle x, z\rangle-\lambda|^{2} \geq 0$ For all $\lambda \in F$
Put $\lambda=\langle x, z\rangle$ because $\langle x, z\rangle \in F, \Rightarrow\|x\|^{2}-|\langle x, z\rangle|^{2} \geq 0 \Rightarrow \mid\langle x, z\rangle\langle\langle\langle | x \|$
Since $z=\frac{y}{\|y\|}$, then $|\langle x, z\rangle| \leq\|x\| \Rightarrow\left|\left\langle x, \frac{y}{\|y\|}\right\rangle\right| \leq\|x\| \Rightarrow \frac{1}{\|y\|}|\langle x, y\rangle| \leq\|x\| \Rightarrow|\langle x, y\rangle| \leq\|x\| \| y$

## Theorem(3.5.6)

Every pre-Hilbert space is normed space

## Proof :

Let $X$ be pre-Hilbert space over the filed $\hat{F}$. Define $\|\cdot\|: X \rightarrow \mathbb{R}$ by $\|x\|=\sqrt{\langle x, x\rangle}$ for all $x \in X$
(1) Since $\langle x, x\rangle \geq 0$ for all $x \in X$, then $\|x\| \geq 0$ for all $x \in X$
(2) $\|x\|=0 \Leftrightarrow \sqrt{\langle x, x\rangle}=0 \Leftrightarrow\langle x, x\rangle=0 \Leftrightarrow x=0$
(3) Let $x \in X$ and let $\lambda \in F \Rightarrow:\|\lambda x\|=\sqrt{\langle\lambda x, \lambda x\rangle}=\sqrt{\lambda \bar{\lambda}\langle x, x\rangle}=|\lambda| \sqrt{\langle x, x\rangle}=|\lambda||x|$
(4) Let $x, y \in X$

Since $\|x+y\|^{2}=\|x\|_{z}^{2}+\langle x, y\rangle+\overline{\langle x, y\rangle}+\|y\|^{2}$ and $\langle x, y\rangle+\overline{\langle x, y\rangle}=2 \operatorname{Re}(\langle x, y\rangle)$, then $\|x+y\|^{2}=\|x\|^{2}+2 \operatorname{Re}(\langle x, y\rangle)+\|y\|^{2}$
Since $\operatorname{Re}(\langle x, y))^{2} \leq\left\langle\langle x, y\rangle \Rightarrow\|x+y\|^{2} \leq\|x\|^{2}+2\right|\langle x, y\rangle \mid+\|y\|^{2}$
Since $|\langle x, y\rangle| \leq\|x\|\|y\|\|x+y\|^{2} \leq\|x\|^{2}+2\|x\| y\|+\| y \|^{2}=\left(\|x\|+\|y\|^{2}\right.$ for all $x, y \in X$.
$\Rightarrow\|x+y\| \leq\|x\|+\|y\|$ for all $x, y \in X \Rightarrow X$ is normed space.

## Theorem(3.5.7)

Let $X$ be a pre-Hilbert space
(1) $\|x+y\|^{2}=\|x\|^{2}+2 \operatorname{Re}\langle x, y\rangle+\|y\|^{2}$ for all $x, y \in X$ (Polar Identity)
(2) $\|x+y\|^{2}+\|x-y\|^{2}=2\|x\|^{2}+2\|y\|^{2}$ for all $x, y \in X$ (Parallelogram Law)

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(3) $\langle x, y\rangle=\frac{1}{4}\left[\|x+y\|^{2}-\|x-y\|^{2}+i\|x+i y\|^{2}-i\|x-i y\|^{2}\right]$ for all $x, y \in X \quad$ (Polarization identity)

## Proof :

(1)
$\|x+y\|^{2}=\langle x+y, x+y\rangle=\langle x, x\rangle+\langle x, y\rangle+\langle y, x\rangle+\langle y, y\rangle=\|x\|^{2}+\langle x, y\rangle+\overline{\langle x, y\rangle}+\|y\|^{2}=\|x\|^{2}+2 \operatorname{Re}\langle x, y\rangle+\|y\|^{2}$
(2) $\|x+y\|^{2}=\langle x+y, x+y\rangle=\|x\|^{2}+\langle x, y\rangle+\langle y, x\rangle+\|y\|^{2}$
$\|x-y\|^{2}=\langle x-y, x-y\rangle=\|x\|^{2}-\langle x, y\rangle-\langle y, x\rangle+\|y\|^{2}$
$\|x+y\|^{2}+\|x-y\|^{2}=2\|x\|^{2}+2\|y\|^{2}$
(3) form (2), (3), we have
$\|x+y\|^{2}-\|x-y\|^{2}=2\langle x, y\rangle+2\langle y, x\rangle, \quad\|x+i y\|^{2}=\|x\|^{2}-i\langle x, y\rangle+i\langle y, x\rangle+\|y\|^{2}$
$\|x-i y\|^{2}=\|x\|^{2}+i\langle x, y\rangle-i\langle y, x\rangle+\|y\|^{2}, \quad i\|x+i y\|^{2}-i\|x-i y\|^{2}=2\langle x, y\rangle-2\langle y, x\rangle$
$\|x+y\|^{2}-\|x-y\|^{2}+i\|x+i y\|^{2}-i\|x-i y\|^{2}=4\langle x, y\rangle$
so that $\quad\langle x, y\rangle=\frac{1}{4}\left[\|x+y\|^{2}-\|x-y\|^{2}+i\|x+i y\|^{2}-i\left\|x-i^{2} y\right\|^{2}\right]$.

## Theorem(3.5.8)

In any be a pre-Hilbert space $X$.
(1) If $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$, then $\left\langle x_{n}, y_{n}\right\rangle \rightarrow\langle x, y\rangle$
(2) If $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ are Cauchy sequence in $X$, then $\left\{\left\langle x_{n}, y_{n}\right\rangle\right\}$ is a Cauchy sequence in $F$

Proof :
(1) $\left.\left\langle x_{n}, y_{n}\right\rangle=\left\langle x+\left(x_{n}-x\right), y+\langle \rangle_{n}-y\right)\right\rangle=\langle x, y\rangle+\left\langle x, y_{n}-y\right\rangle+\left\langle x_{n}-x, y\right\rangle+\left\langle x_{n}-x, y_{n}-y\right\rangle$
$\left\langle x_{n}, y_{n}\right\rangle-\langle x, y\rangle=\left\langle x, y_{n}-y\right\rangle+\left\langle x_{n}-x, y\right\rangle+\left\langle x_{n}-x, y_{n}-y\right\rangle$
$\left|\left\langle x_{n}, y_{n}\right\rangle-\langle x, y\rangle\right|=\left|\left\langle x, y_{n}-y\right\rangle+\left\langle x_{n}-x, y\right\rangle+\left\langle x_{n}-x, y_{n}-y\right\rangle\right| \leq\left|\left\langle x, y_{n}-y\right\rangle\right|+\left|\left\langle x_{n}-x, y\right\rangle\right|+\left|\left\langle x_{n}-x, y_{n}-y\right\rangle\right|$
$\left|\left\langle x_{n}, y_{n}\right\rangle-\langle x, y\rangle\right| \leq\|x\|\left\|y_{n}-y\right\|+\left\|x_{n}-x\right\|\|y\|+\left\|x_{n}-x\right\|\left\|y_{n}-y\right\|$
Since $x_{n} \rightarrow x$, then $\left\|x_{n}-x\right\| \rightarrow 0$ as $n \rightarrow \infty$, also Since $y_{n} \rightarrow y$, then $\left\|y_{n}-y\right\| \rightarrow 0$ as $n \rightarrow \infty$
$\Rightarrow\left|\left\langle x_{n}, y_{n}\right\rangle-\langle x, y\rangle\right| \rightarrow 0$ as $n \rightarrow \infty$
(2) Similarly, we have $\left|\left\langle x_{n}, y_{n}\right\rangle-\left\langle x_{m}, y_{m}\right\rangle\right| \leq\left\|x_{m}\right\|\left\|y_{n}-y_{m}\right\|+\left\|x_{n}-x_{m}\right\|\left\|y_{m}\right\|+\left\|x_{n}-x_{m}\right\|\left\|y_{n}-y_{m}\right\|$

Since $\left\|x_{n}\right\|$ and $\left\|y_{m}\right\|$ are bounded $\Rightarrow\left|\left\langle x_{n}, y_{n}\right\rangle-\left\langle x_{m}, y_{m}\right\rangle\right| \rightarrow 0$ as $n, m \rightarrow \infty$

## Corollary(3.5.9)

In any be a pre-Hilbert space $X$.
(1) If $x_{n} \rightarrow x$, then $\left\|x_{n}\right\| \rightarrow\|x\|$
(2) If $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$, then $\left\{\left\|x_{n}\right\|\right\}$ converge.
(3) $\langle$,$\rangle is (jointly) continuous.$

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## Definition (3.5.10)

A complete pre-Hilbert space is called a Hilbert space, i.e. A Hilbert space is a vector space $X$ over a filed $F$ together with an inner product $\langle$,$\rangle such that relative to the$ metric $d(x, y)=\|x-y\|$ induced by the norm, $X$ is a complete metric space.

## Example (3.5.11)

(1) The space $\mathbb{R}^{n}$ is a Hilbert space with inner product defined by $\langle x, y\rangle=\sum_{i=1}^{n} x_{i} y_{i}$ "where $x=\left(x_{1}, x_{2}, \cdots, x_{n}\right), \quad y=\left(y_{1}, y_{2}, \cdots, y_{n}\right) \in \mathbb{R}^{n}$
(2) The space $\ell^{2}$ is a Hilbert space with inner product defined by $\langle x, y\rangle=\sum_{i=1}^{\infty} x_{i} \overline{y_{i}}$ where $x=\left(x_{1}, x_{2}, \cdots\right), \quad y=\left(y_{1}, y_{2}, \cdots\right) \in \ell^{2}$ satisfying the convergence $\sum_{i=1}^{\infty}\left|x_{i}\right\rangle^{2}<\infty$.

## Example (3.5.12)

Let $X=C[-1,1]$.Define $\langle\rangle:, X \times X \rightarrow F$ by $\langle f, g\rangle=\int_{-1}^{1} f(x) g(x) d x$ for all $f,, g \in X$. Then $X$ is not Hilbert space
Ans:
We now show that $X$ is not complete, Consider the sequence $\left\{f_{n}\right\}$ in $X$ defined as follows
$f_{n}(x)=\left\{\begin{array}{lc}0, & -1 \leq x \leq 0 \\ n x, & 0<x<\frac{1}{n} \\ 1, & \frac{1}{n} \leq x \leq 1\end{array}\right.$
$\left\|f_{n}-f_{m}\right\|=\left\langle f_{n}-f_{m}, f_{n}-f_{m}\right\rangle=\frac{(n-m)^{2}}{3 n^{2} m} \Rightarrow\left\|f_{m}-f_{n}\right\| \rightarrow 0$ as $n, m \rightarrow \infty$
$\Rightarrow\left\{f_{n}\right\}$ is a Cauchy sequence. But this sequence is not convergent.
For if there existed a $f \in X$ such that $f_{n} \rightarrow f \Rightarrow$

$$
f(x)=\lim _{n \rightarrow \infty} f_{n}(x)=\left\{\begin{array}{cc}
1, & -1 \leq x \leq 0 \\
0, & 0<x \leq 1
\end{array}\right.
$$

This contradiction, because $f$ is not continuous.

## Example (3.5.13)

Every Hilbert space is a Banach space, but the converse is not true Ans:

Let $X$ be a Hilbert space, then $X$ is complete Pre-Hilbert space

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Since every Pre-Hilbert space is normed space, then $X$ is complete normed space. Hence $X$ is Banach space.
The converse : take $X=\ell^{p}, \quad p \neq 2$, then $X$ is a Banach space but not Hilbert space, because $X$ is not Pre-Hilbert space, i.e.
Let $x=(1,-1,0,0, \cdots), \quad y=(1,1,0,0, \cdots) \Rightarrow x, y \in \ell^{p}$
$\|x\|=\|y\|=2^{\frac{1}{p}},\|x+y\|=\|x-y\|=2$, so that $\|x+y\|^{2}+\|x-y\|^{2} \neq 2\|x\|^{2}+2\|y\|^{2}$

## Theorem (3.5.14)

A closed convex subset $A$ of a Hilbert space $X$ contains a unique element of smallest norm.
Proof:
Let $d=\inf \{\|x\|: x \in A\}$, by definition of the $d$ there is a sequence $\left\{x_{n}\right\}$ in $A$ such that $\left\|x_{n}\right\| \rightarrow d$. To prove that $\left\{x_{n}\right\}$ a Cauchy sequence in $A$, i.e. to prove $\left\|x_{n}-x_{m}\right\| \rightarrow 0$ as $n, m \rightarrow \infty$, Consider the two vectors $x_{m}$ and $x_{n}$ belonging to the sequence $\left\{x_{n}\right\}$
Since $A$ is convex and $x_{n}, x_{m} \in A \Rightarrow \frac{1}{2}\left(x_{n}+x_{m}\right) \in A$
Hence by the definition of $d$, we have $\left\|\frac{1}{2}\left(x_{n}+x_{m}\right)\right\| \geq d$ so that $\left\|x_{n}+x_{m}\right\| \geq 2 d$
Since $\left\|x_{n}+x_{m}\right\|^{2}+\left\|x_{n}-x_{m}\right\|^{2}=2\left\|x_{n}\right\|^{2}+2\left\|x_{m}\right\|^{2}$
$\Rightarrow\left\|x_{n}-x_{m}\right\|^{2}=2\left\|x_{n}\right\|^{2}+2\left\|x_{m}\right\|^{2}-\left.\left\|x_{n}+\left.x_{m}\right|^{2} \leq 2\right\| x_{n}\right|^{2}+2 \|\left. x_{m}\right|^{2}-4 d^{2}$
Now $\left\|x_{n}\right\| \rightarrow d$ and $\left\|x_{m}\right\| \rightarrow d$ as $n, m \rightarrow \infty$. Therefore
$2\left\|x_{n}\right\|^{2}+2\left\|x_{m}\right\|^{2}-4 d^{2} \rightarrow 2 d^{2}+2 d^{2}-4 d^{2} \geqslant 0$
Hence as $n, m \rightarrow \infty$, we have $\left\|x_{n},-x_{m}\right\| \rightarrow 0$. Therefore $\left\{x_{n}\right\}$ is a Cauchy sequence in $A$
Since $X$ is complete, then there exists $x \in X$ such that $x_{n} \rightarrow x$
Since $\left\{x_{n}\right\}$ in $A$, then $\bar{x} \in \bar{A}$, but $A$ is closed, then $\bar{A}=A \Rightarrow x \in A$
$x_{n} \rightarrow x \Rightarrow\left\|x_{n}\right\| \rightarrow\|x\| \Rightarrow\|x\|=d$
Suppose $x$ and $y$ are two elements in $A$ such that $\|x\|=d$ and $\|y\|=d$
Since $A$ is convex $\Rightarrow \frac{1}{2}(x+y) \in A \Rightarrow\left\|\frac{1}{2}(x+y)\right\| \geq d$ so that $\|x+y\| \geq 2 d$
Now
$\left\|x-y\left|\left\|^{2}=2\right\| x\left\|^{2}+\right\| y\right|^{2}-\right\| x+y\left\|^{2} \leq 2 d^{2}+2 d^{2}-4 d^{2}=0 \quad \Rightarrow \quad\right\| x-y \|^{2} \leq 0$
Since $\|x-y\|^{2} \geq 0 \Rightarrow\|x-y\|^{2}=0 \Rightarrow\|x-y\|=0 \Rightarrow x-y=0 \Rightarrow x=y$
Theorem (3.5.15)
Let $A$ be a non-empty closed convex subset of a Hilbert space $X$ and let $x_{0} \notin A$. Then there exists a unique element $a \in A$ such that $\left\|x_{0}-a\right\|=d\left(x_{0}, A\right)$.

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## Proof:

$$
d\left(x_{0}, A\right)=\inf \left\{d\left(x_{0}, x\right): x \in A\right\}=\inf \left\{\left\|x_{0}-x\right\|: x \in A\right\},
$$

Let $d=d\left(x_{0}, A\right)$ by definition of the infimum there is a sequence $\left\{x_{n}\right\}$ in $A$ such that $\left\|x_{0}-x_{n}\right\| \rightarrow d$. To prove that $\left\{x_{n}\right\}$ a Cauchy sequence in $A$, i.e. to prove $\left\|x_{n}-x_{m}\right\| \rightarrow 0$ as $n, m \rightarrow \infty$, Consider the two vectors $x_{m}$ and $x_{n}$ belonging to the sequence $\left\{x_{n}\right\}$
Since $A$ is convex and $x_{n}, x_{m} \in A \Rightarrow \frac{1}{2}\left(x_{n}+x_{m}\right) \in A$
Hence by the definition of $d$, we have $\left\|x_{0}-\frac{1}{2}\left(x_{n}+x_{m}\right)\right\| \geq d$ so that $\left\|2 x_{0}-\left(x_{n}+x_{m}\right)\right\| \geq 2 d$

$$
\begin{align*}
\left\|x_{n}-x_{m}\right\|^{2} & =\left\|\left(x_{0}-x_{n}\right)-\left(x_{0}-x_{m}\right)\right\|^{2}=2\left\|x_{0}-x_{n}\right\|^{2}+2\left\|x_{0}-x_{m}\right\|^{2}-\left\|\left(x_{0}-x_{n}\right)+\left(x_{0}-x_{m}\right)\right\|^{2}  \tag{1}\\
& =2\left\|x_{0}-x_{n}\right\|^{2}+2\left\|x_{0}-x_{m}\right\|^{2}-\left\|2 x_{0}-\left(x_{n}+x_{m}\right)\right\|^{2}
\end{align*}
$$

Since $\left\|x_{0}-x_{n}\right\| \rightarrow d$ and $\left\|x_{0}-x_{m}\right\| \rightarrow d$ as $n, m \rightarrow \infty$. Therefore $\left\|x_{n}=x_{m}\right\| \rightarrow 0$ as $n, m \rightarrow \infty$.
Hence $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$
Since $X$ is complete, then there exists $a \in X$ such that $X_{n} \rightarrow a$
Since $\left\{x_{n}\right\}$ in $A$, then $a \in \bar{A}$, but $A$ is closed, then $\vec{A}=A \Rightarrow a \in A$
$x_{n} \rightarrow x \Rightarrow\left\|x_{0}-x_{n}\right\| \rightarrow\left\|x_{0}-a\right\|=d$
Suppose $a$ and $b$ are two elements in $A$ such that $\left\|x_{0}-a\right\|=d$ and $\left\|x_{0}-b\right\|=d$
Now

$$
\begin{aligned}
& \|a-b\|^{2}=\left\|\left(x_{0}-b\right)-\left(x_{0}-a\right)\right\|^{2}=2\left\|x_{0}-b\right\|^{2}+2\left\|x_{0}-a\right\|^{2}-\left\|\left(x_{0}-b\right)+\left(x_{0}-a\right)\right\|^{2} \leq 2 d^{2}+2 d^{2}-\left\|2 x_{0}-(a+b)\right\|^{2} \leq 0 \\
& \quad \Rightarrow\|a-b\|^{2} \leq 0
\end{aligned}
$$

Since $\|a-b\|^{2} \geq 0 \Rightarrow\|a-b\|^{2}=\theta^{2} \Rightarrow\|a-b\|=0 \Rightarrow a-b=0 \Rightarrow a=b$

## Orthogonality :

## Definition(3.5.16)

Let $X$ be a pre-Hilbert space, and $x, y \in X$. we say that $x$ is orthogonal to $y$ (written $x \perp y$ ), if $\langle x, y\rangle=0$

## Remarks

(1) The relation of orthogonality is symmetric. i.e. if $x \perp y$, then $y \perp x$
since if $x \perp y$, then $\langle x, y\rangle=0 \Rightarrow \overline{\langle x, y\rangle}=\overline{0}=0 \Rightarrow\langle y, x\rangle=0 \Rightarrow y \perp x$
(2) If $x \perp y$, then $\lambda x \perp y$ for all $\lambda \in F$

Since $x \perp y$, then $\langle x, y\rangle=0$, and $\langle\lambda x, y\rangle=\lambda\langle x, y\rangle=\lambda \cdot 0=0 \Rightarrow \lambda x \perp y$
(3) The zero vector is orthogonal to every vector.

For, every vector $x$ in $X$, we have $\langle 0, x\rangle=0$. Therefore $0 \perp x$ for all $x \in X$
(4) The zero vector is the only vector which is orthogonal to itself.

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$$
\text { If } x \perp x \Rightarrow\langle x, x\rangle=0 \Rightarrow x=0
$$

## Example (3.5.17)

If $x$ is orthogonal to each of $x_{1}, x_{2}, \cdots, x_{n}$, then $x$ is orthogonal to every linear combination of this $x_{k}$

## Proof:

Let $x \perp x_{i}$ for all $i=1,2, \cdots, n \Rightarrow\left\langle x, x_{i}\right\rangle=0$ for all $i=1,2, \cdots, n$
Let $z=\sum_{i=1}^{n} \lambda_{i} x_{i}$ such that $\lambda_{i} \in F$ for all $i=1,2, \cdots, n$
$\langle x, z\rangle=\left\langle x, \sum_{i=1}^{n} \lambda_{i} x_{i}\right\rangle=\sum_{i=1}^{n} \bar{\lambda}_{i}\left\langle x, x_{i}\right\rangle=0 \Rightarrow x \perp z$.

## Theorem (3.5.18)

If $x$ and $y$ are any two orthogonal vectors in a pre-Hilbert space $X$, then

## Proof :

$$
\|x+y\|^{2}=\|x-y\|^{2}=\|x\|^{2}+\|y\|^{2}
$$

$$
\begin{aligned}
& \quad \text { Since } x \perp y \Rightarrow\langle x, y\rangle=\langle y, x\rangle=0 \text {, so } \\
& \|x+y\|^{2}=\langle x+y, x+y\rangle=\langle x, x\rangle+\langle x, y\rangle+\langle y, x\rangle+\langle y, y\rangle=\|x\|^{2}+0+0+\|y\|^{2}=\|x\|^{2}+\|y\|^{2} \\
& \text { Similarly }\|x-y\|^{2}=\|x\|^{2}+\|y\|^{7} \text {. }
\end{aligned}
$$

## Corollary (3.5.19)

If $x_{1}, \cdots, x_{n} x_{1}, \cdots, x_{n}$ are orthogonal vectors space in a pre-Hilbert space $X$ then $\left\|\sum_{i=1}^{n} x_{i}\right\|^{2}=\sum_{i=1}^{n}\left\|x_{i}\right\|^{2}$

## Definition(3.5.20)

Let $A$ be a non-empty subsets of a pre-Hilbert space $X$. A vector $x \in X$ is said to be orthogonal to $A$ (written $x \perp A$ ), if $x \perp y$ for all $y \in A$.
To non-empty subsets $A$ and $B$ of a pre-Hilbert space $X$ are said to be orthogonal (written $A \perp B$ ), if $x \perp y$ for all $x \in A$ and for all $y \in B$.

## Remark

If $M_{1}$ and $M_{2}$ are subspaces of a pre-Hilbert space $X$ such that $M_{1} \perp M_{2}$, then $M_{1} \cap M_{2}=\{0\}$

## Definition(3.5.21)

Let $A$ be a non-empty subsets of a pre-Hilbert space $X$. The orthogonal complement of $A$ , written as $A^{\perp}$ and is defined by $A^{\perp}=\{x \in X: x \perp y, \forall y \in A\}=\{x \in X: x \perp A\}$.
Thus $A^{\perp}$ is the set of all elements in $X$ which are orthogonal to every elements in $A$.
And we define $\left(A^{\perp}\right)^{\perp}$ written as $A^{\perp \perp}$, by $A^{\perp \perp}=\left\{x \in X: x \perp y, \forall y \in A^{\perp}\right\}$

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## Remarks

Let $X$ be a pre-Hilbert space, then $X^{\perp}=\{0\}$ and $\{0\}^{\perp}=X$

## Theorem (3.5.22)

Let $A$ and $B$ be subsets of a pre-Hilbert space $X$, then
(1) $A \cap A^{\perp} \subset\{0\}$, and
$A \subseteq A^{\perp \perp}$
(2) if $A \subset B$, then $B^{\perp} \subset A^{\perp}$

## Proof :

(1) Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp} \Rightarrow\langle x, x\rangle=0 \Rightarrow x=0$
(2) Let $x \in B^{\perp} \Rightarrow\langle x, y\rangle=0$ for all $y \in B$

Since $A \subset B \Rightarrow\langle x, y\rangle=0$ for all $y \in A \Rightarrow x \in A^{\perp} \Rightarrow B^{\perp} \subset A^{\perp}$

## Theorem (3.5.23)

Let $A$ be a non-empty subsets of a pre-Hilbert space $X$. Then $A$ is closed subspace of $X$.

## Proof:

First to prove $A^{\perp}$ is a subspace
Since $0 \perp x$ for all $x \in A \Rightarrow 0 \in A^{\perp} \Rightarrow A^{\perp} \neq \phi$
Let $x, y \in A^{\perp}, \quad \alpha, \beta \in F$, then $\langle x, z\rangle=0$ and $\langle y, z\rangle=0$ for all $z \in A$,
For all $z \in A$, we have $\langle\alpha x+\beta y, z\rangle=\alpha\langle x, z\rangle+\beta\langle y, z\rangle=\alpha(0)+\beta(0)=0$
$\Rightarrow \alpha x+\beta y \in A^{\perp} \Rightarrow A^{\perp}$ is a subspace of $X$.
Now we shall show that $A^{\perp}$ is a closed subset of $X$.(i.e. to prove $\overline{A^{\perp}}=A^{\perp}$ )
Let $x \in \overline{A^{\perp}} \Rightarrow$ there is a sequence $\left\{x_{n}\right\}$ in $A^{\perp}$ such that $x_{n} \rightarrow x$.
Let $y \in A \Rightarrow\left\langle x_{n}, y\right\rangle=0$ for all $n \in \mathbb{Z}^{+}\left(x_{n} \in A^{\perp}\right.$ for all $\left.n \in \mathbb{Z}^{+}\right)$
Since $x_{n} \rightarrow x \Rightarrow\left\langle x_{n}, y\right\rangle \rightarrow\langle x, y\rangle \Rightarrow\langle x, y\rangle=0$
For all $y \in A \Rightarrow\langle x, y\rangle=0 \Rightarrow x \in A^{\perp} \Rightarrow \overline{A^{\perp}} \subseteq A^{\perp}$, but $A^{\perp} \subseteq \overline{A^{\perp}}$, then $\overline{A^{\perp}}=A^{\perp}$
So that $A^{\perp}$ is closed subspace of $X$.

## Theorem (3.5.24)

Let $M$ be a proper closed subspace of a Hilbert space $X$, then there exists a non-zero element $x_{0} \in X$ such that $x_{0} \perp M$.

## Proof :

Since $M$ is a proper subspace of , then there is $y \in X$ such that $y \notin M$
Let $d=d(y, M) \Rightarrow d=\inf \{\|y-x\|: x \in M\}$
Since $y \notin M \Rightarrow y \neq x$ for all $x \in M \quad \Rightarrow d>0$
By theorem (3.5.15), there is $x \in M$ such that $\|y-x\|=d$
Put $x_{0}=y-x \quad \Rightarrow x_{0} \in X \quad \Rightarrow \quad\left\|x_{0}\right\|=\|y-x\|=d>0$
Therefore $x_{0}$ is a non-zero element in $X$.

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Now we shall show that $x_{0} \perp M$, (i.e. $x_{0} \perp m$ for all $m \in M$ )
Let $m \in M$, for any $\lambda \in F$, we have $x_{0}-\lambda m=y-x-\lambda m=y-(x+\lambda m)$
Since $x, m \in M$ and $M$ is a subspace, then $x+\lambda m \in M$. Then by definition of $d$, we
have $\left.\left.\|y-(x+\lambda m)\| \geq d \Rightarrow \| x_{0}-\lambda m\right)\|\geq d=\| x_{0}\|\Rightarrow\| x_{0}-\lambda m\right)\left\|^{2} \geq\right\| x_{0} \|^{2}$
$\Rightarrow\left\langle x_{0}-\lambda m, x_{0}-\lambda m\right\rangle \geq\left\langle x_{0}, x_{0}\right\rangle \Rightarrow\left\langle x_{0}, x_{0}\right\rangle-\bar{\lambda}\left\langle x_{0}, m\right\rangle-\lambda\left\langle m, x_{0}\right\rangle+\lambda \bar{\lambda}\langle m, m\rangle \geq\left\langle x_{0}, x_{0}\right\rangle$,
$\Rightarrow-\bar{\lambda}\left\langle x_{0}, m\right\rangle-\lambda\left\langle m, x_{0}\right\rangle+\lambda \bar{\lambda}\langle m, m\rangle \geq 0 \Rightarrow-\bar{\lambda}\left\langle x_{0}, m\right\rangle-\lambda \overline{\left\langle x_{0}, m\right\rangle}+\lambda \bar{\lambda}\langle m, m\rangle \geq 0$ for all $\lambda \in F$
Let us take $\lambda=\beta\left\langle x_{0}, m\right\rangle$ where $\beta$ is an arbitrary real number. Then $\bar{\lambda}=\overline{\beta\left\langle x_{0}^{0}, m\right\rangle}=\beta \overline{\left\langle x_{0}, m\right\rangle}$
$\Rightarrow-\boldsymbol{\beta} \overline{\left\langle x_{0}, m\right\rangle}\left\langle x_{0}, m\right\rangle-\beta\left\langle x_{0}, m\right\rangle \overline{\left\langle x_{0}, m\right\rangle}+\beta^{2}\left\langle x_{0}, m\right\rangle \overline{\left\langle x_{0}, m\right\rangle}\|m\|^{2} \geq 0$
$\Rightarrow-2 \beta\left|\left\langle x_{0}, m\right\rangle\right|^{2}+\beta^{2}\left|\left\langle x_{0}, m\right\rangle\right|^{2}\|m\|^{2} \geq 0 \Rightarrow \beta\left|\left\langle x_{0}, m\right\rangle\right|^{2}\left(\beta\|m\|^{2}-2\right) \geq 0$
(1) for all $\beta \in \mathbb{R}$

Suppose that $\left\langle x_{0}, m\right\rangle \neq 0$. Then taking $\beta$ positive and so small that $\beta\|m\|^{2}<2$, we see from (1) that $\beta\left|\left\langle x_{0}, m\right\rangle\right|^{2}\left(\beta\|m\|^{2}-2\right)<0$. This contradiction (1). Hence we must $\left\langle x_{0}, m\right\rangle=0 \Rightarrow x_{0} \perp m$ for all $m \in M \Rightarrow x_{0} \perp M$

## Theorem (3.5.25)

If $M$ is a subspace of a Hilbert space $X$, then $M$ is closed iff $M=M^{\perp \perp}$.
Proof :
Assume $M=M^{\perp \perp}$
Since $\left(M^{\perp}\right)^{\perp}$ is closed subspace of $X$ and $M^{\Perp}=\left(M^{\perp}\right)^{\perp} \Rightarrow M^{\Perp}$ is closed subspace of $X$
Since $M=M^{\perp \perp} \Rightarrow M$ is closed subspace of $X$
Conversely suppose that $M$ is closed subspace of $X$
By theorem(3.5.22), we have $M \subset M^{+1}$.
Suppose now that this inclusion is proper, i.e. $M \neq M^{\perp \perp}$
Now $M^{\Perp}$ is a Hilbert space and $M$ is a proper closed subspace of $M^{\Perp}$, by theorem (3.5.24), there exists a non-zero $x_{0} \in M^{\Perp}$ such that $x_{0} \perp M \Rightarrow x_{0} \in M^{\perp} \Rightarrow x_{0} \in M^{\perp} \cap M^{\Perp}$
Since $M^{\perp}$ is a subspace of $X$, therefore by theorem(4.22), we have $M^{\perp} \cap M^{\Perp}=\{0\}$
From (1) and (2) we conclude that $x_{0}=0$.This contradicts that $x_{0}$ is a non-zero element.
Therefore the inclusion $M \subset M^{\perp \perp}$ cannot be proper and we must have $M=M^{\perp \perp}$.

## Theorem (3.5.26)

If $M_{1}$ and $M_{2}$ are closed subspaces of a Hilbert space $X$ such that $M_{1} \perp M_{2}$, then the subspace $M_{1}+M_{2}$ is also closed.

## Proof :

Let $z \in \overline{M_{1}+M_{2}}$, then there exists a sequence $\left\{z_{n}\right\}$ in $M_{1}+M_{2}$ such that $z_{n} \rightarrow z$

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Since $M_{1} \perp M_{2} \Rightarrow M_{1} \cap M_{2}=\{0\} \Rightarrow$ the subspace $M_{1}+M_{2}$ is direct sum of the subspaces $M_{1}$ and $M_{2}$, i.e. $M_{1}+M_{2}=M_{1} \oplus M_{2} . \Rightarrow$ each $z_{n}$ can be uniquely written as $z_{n}=x_{n}+y_{n}$ where $x_{n} \in M_{1}, \quad y_{n} \in M_{2}$
Consider $z_{n}=x_{n}+y_{n}, \quad z_{m}=x_{m}+y_{m} \Rightarrow z_{n}-z_{m}=\left(x_{n}-x_{m}\right)+\left(y_{n}-y_{m}\right)$
Since $x_{n}-x_{m} \in M_{1}, \quad y_{n}-y_{m} \in M_{2}$ and $M_{1} \perp M_{2} \Rightarrow\left(x_{n}-x_{m}\right) \perp\left(y_{n}-y_{m}\right)$
$\Rightarrow\left\|\left(x_{n}-x_{m}\right)+\left(y_{n}-y_{m}\right)\right\|^{2}=\left\|x_{n}-x_{m}\right\|^{2}+\left\|y_{n}-y_{m}\right\|^{2} \Rightarrow\left\|z_{n}-z_{m}\right\|^{2}=\left\|x_{n}-x_{m}\right\|^{2}+\left\|y_{n}-y_{m}\right\|^{2}$
Since $z_{n} \rightarrow z \Rightarrow\left\{z_{n}\right\}$ is a Cauchy sequence $\Rightarrow\left\|z_{n}-z_{m}\right\|^{2} \rightarrow 0$ as $n, m \rightarrow \infty$
$\Rightarrow\left\|x_{n}-x_{m}\right\|^{2}+\left\|y_{n}-y_{m}\right\|^{2} \rightarrow 0$ as $n, m \rightarrow \infty \Rightarrow\left\|x_{n}-x_{m}\right\|^{2} \rightarrow 0$ and $\left\|y_{n}-y_{m}\right\|^{2} \rightarrow 0$ as $n, m \rightarrow \infty$
$\Rightarrow\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ are Cauchy sequence in $M_{1}$ and $M_{2}$ respectively.
Since $X$ is complete and $M_{1}, M_{2}$ are closed subspace of $X$, therefore $M_{1}, M_{2}$ are complete.
$\Rightarrow\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ are convergent sequence in $M_{1}$ and $M_{2}$ respectively. Therefore there
exists $x \in M_{1}$ and $y \in M_{2}$ such that $x_{n} \rightarrow x$ and $y_{n} \rightarrow y \Rightarrow z_{n}=x_{n}+y_{n} \rightarrow x+y \in M_{1}+M_{2}$
Since $z_{n} \rightarrow z \quad z=x+y \quad \Rightarrow \quad z \in M_{1}+M_{2} \Rightarrow \overline{M_{1}+M_{2}} \subseteq M_{1}+M_{2}$, but
$M_{1}+M_{2} \subseteq \overline{M_{1}+M_{2}}$, then $\overline{M_{1}+M_{2}}=M_{1}+M_{2}$. Therefore $M_{1}+M_{2}$ is a closed.
Theorem (3.5.27) Projection Theorem
If $M$ is closed subspace of a Hilbert space $X$, then $X=M \oplus M^{\perp}$
Proof:
Since $M$ and $M^{\perp}$ are closed subspaces of $X$ and $M \perp M^{\perp}$, then $M \cap M^{\perp}=\{0\}$
In order to show that $X=M \oplus M^{\perp}$ we need only check that $X=M+M^{\perp}$
Let us put $Y=M+M^{\perp} \Rightarrow M \subset Y$ and $M^{\perp} \subset Y \Rightarrow Y^{\perp} \subset M^{\perp}$ and $Y^{\perp} \subset M^{\perp}$
$\Rightarrow \quad Y^{\perp} \subset M^{\perp} \cap M^{\perp}$
Since $M$ is closed subspace of $X$, then by theorem(3.5.25), we have $M=M^{\perp}$
$\Rightarrow Y^{\perp} \subset M^{\perp} \cap M$
But $M \cap M^{\perp}=\{0\} \Rightarrow Y^{\perp} \subset\{0\} \Rightarrow Y^{\perp}=\{0\} \Rightarrow Y^{\perp}=\{0\}^{\perp}=X$
Since $M$ and $M^{-\perp}$ are closed subspaces of $X$ and $M \perp M^{\perp}$, by theorem(3.5.26), we have $M+M^{\perp}$ is closed subspace of $X \Rightarrow Y$ is closed subspace of $X$, by theorem(3.5.25), we have $Y=Y^{\perp} \stackrel{\Rightarrow}{\Rightarrow} Y=X$.

## Orthonormal sets

## Definitions(3.5.28)

A subsets $A$ of a pre-Hilbert space $X$ is said to be
(1) Orthogonal set if $x \perp y$ for all $x, y \in A$ and $x \neq y$
(2) Orthonormal set if $A$ is orthogonal and $\|x\|=1$ for all $x \in A$, i.e. $A$ is Orthonormal set

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if

$$
\langle x, y\rangle= \begin{cases}0, & x \neq y \\ 1, & x=y\end{cases}
$$

for all $x, y \in A$. In special case, a sequence $\left\{x_{n}\right\}$ is called an Orthogonal sequence in $X$ if $x_{n} \perp x_{m}$ for all $n \neq m$, and $\left\{x_{n}\right\}$ is called an Orthonormal sequence in $X$ if $\left\langle x_{n}, x_{m}\right\rangle= \begin{cases}0, & n \neq m \\ 1, & n=m\end{cases}$

## Example(3.5.29)

Let $X=\mathbb{R}^{3}$ and $A=\{(1,2,2),(2,1,-2),(2,-2,1)\}$, then $A$ is an orthogonal set under the usual inner product
Ans:
Let $x=(1,2,2), \quad y=(2,1,-2), \quad z=(2,-2,1)$
$\langle x, y\rangle=\sum_{i=1}^{3} x_{i} y_{i}=1 \times 2+2 \times 1+2 \times(-2)=2+2-4=0$,
$\langle x, z\rangle=\sum_{i=1}^{3} x_{i} z_{i}=1 \times 2+2 \times(-2)+2 \times 1=2-4+2=0$
$\langle y, z\rangle=\sum_{i=1}^{3} y_{i} z_{i}=2 \times 2+1 \times(-2)+(-2) \times 1=4-2-2=0$

## Remark

(1) An Orthonormal set cannot contain zero vector because $\|0\|=0 \neq 1$
(2) Every pre-Hilbert space $X$ which is not equal to zero space possesses on Orthonormal set.
Let $0 \neq x \in X \Rightarrow\|\mid\| \| \neq 0$. Put $y=\frac{x}{\|x\|} \Rightarrow\|y\|=1 \Rightarrow\{y\}$ is Orthonormal set.

## Theorem(3.5.30)

Let $A$ be an-orthogonal set in pre-Hilbert space $X$, then
(1) The set $A$ is linearly independent
(2) The set $B=\left\{\frac{x}{\|x\|}: x \in A\right\}$ is Orthonormal set.

## Proof:

(1) Let $\left\{x_{1}, \cdots, x_{n}\right\}$ be a finite subset of $A \Rightarrow \sum_{i=1}^{m} \lambda_{i} x_{i}=0 \Rightarrow\left\|\sum_{i=1}^{m} \lambda_{i} x_{i}\right\|=\|0\|=0$

Since $x_{i} \perp x_{j}$ for all $i \neq j, \Rightarrow \sum_{i=1}^{n}\left|\lambda_{i}\right|^{2}\left\|x_{i}\right\|^{2}=0$

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Since $x_{i} \neq 0$, then $\left\|x_{i}\right\|>0$ for all $i$, so $\lambda_{i}=0$.

## Theorem(3.5.31)

Let $A=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ be a finite orthonormal set in pre-Hilbert space $X$. If $x$ is any vector in $X$, then
(1) $\left\|x-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle x_{i}\right\|^{2}=\|x\|^{2}-\left.\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle\right|^{2}$
(2) $\sum_{i=1}^{n}\left|\left\langle x, x_{i}\right\rangle\right|^{2} \leq\|x\|^{2}$
(3) $\left(x-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle x_{i}\right) \perp x$ each $j$

## Proof :

(1) Let $\lambda_{i}=\left\langle x, x_{i}\right\rangle$
$\left\|x-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle x_{i}\right\|^{2}=\left\|x-\sum_{i=1}^{n} \lambda_{i} x_{i}\right\|^{2}=\left\langle x-\sum_{i=1}^{n} \lambda_{i} x_{i}, x-\sum_{i 1}^{n} \lambda_{i} x_{i}\right\rangle$
$\left.=\langle x, x\rangle-\left\langle x, \sum_{i=1}^{n} \lambda_{i} x_{i}\right\rangle-\left\langle\sum_{i=1}^{n} \lambda_{i} x_{i}, x\right\rangle+\left\langle\sum_{i=1}^{n} \lambda_{i} x_{i}, \sum_{i=1}^{n} \lambda_{i} x_{i}\right\rangle=\|x\|^{2}-\sum_{i=1}^{n} \lambda_{i}\left\langle x, x_{j}\right\rangle\right\rangle-\sum_{i=1}^{n} \lambda_{i}\left\langle x_{i}, x\right\rangle+\left\|\sum_{i=1}^{n} \lambda_{i} x_{i}\right\|^{2}$
$=\|x\|^{2}-\sum_{i=1}^{n} \overline{\lambda_{i}} \lambda_{i}-\sum_{i=1}^{n} \lambda_{i} \bar{\lambda}_{i}+\left.\sum_{i=1}^{n}\left|\lambda_{i}\right|^{2}\left|x_{i}\left\|^{2}=\right\| x\left\|^{2}-\sum_{i=1}^{n}\left|\lambda_{i}\right|^{2}-\sum_{i=1}^{n}\left|\lambda_{i}\right|^{2}+\sum_{i=1}^{n}\left|\lambda_{i}\right|^{2}=\right\| x\left\|^{2}-\sum_{i=1}^{n}\left|\lambda_{i}\right|^{2}=\right\| x \|^{2}-\sum_{i=1}^{n}\right|\left\langle x, x_{i}\right\rangle\right|^{2}$
(2) Since $\left\|x-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle x_{i}\right\|^{2} \geq 0 \Rightarrow\|x\|^{2}-\sum_{i=1}^{n}\left|\left\langle x, x_{i}\right\rangle\right\rangle^{2} \geq 0 \Rightarrow \sum_{i=1}^{n}\left|\left\langle x, x_{i}\right\rangle\right|^{2} \leq\|x\|^{2}$
(3) $\left\langle x-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle x_{i}, x_{j}\right\rangle=\left\langle x, x_{j}\right\rangle-\left\langle\sum_{i=1}^{n}\left\langle x, x_{i}^{s}\left\langle x_{i}, x_{j}\right\rangle=\left\langle x, x_{j}\right\rangle-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle\left\langle x_{i}, x_{j}\right\rangle\right.\right.$

Since $\left\langle x_{i}, x_{j}\right\rangle= \begin{cases}0, & i \neq j \\ 1 & i=j\end{cases}$
$\left\langle x-\sum_{i=1}^{n}\left\langle x, x_{i} \mid x_{i}, x_{j}\right\rangle=\left\langle x, x_{j}\right\rangle-\left\langle x_{x}, x_{j}\right\rangle=0\right.$. Hence $\quad\left(x-\sum_{i=1}^{n}\left\langle x, x_{i}\right\rangle x_{i}\right) \perp x_{j}$ for each $j$.
Corollary (3.5.32) Bessel's Inequality
If $\left\{x_{n}\right\}$ is an orthonormal sequence in pre-Hilbert space $X$, then $\sum_{i=1}^{\infty}\left|\left\langle x, x_{i}\right\rangle\right|^{2} \leq\|x\|^{2}$ for all $x \in X$.
Theorem(3.5.33) Gram-Schmidt Theorem
If $\left\{y_{n}\right\}$ is dinearly independent in pre-Hilbert space $X$, then there exists an orthonormal sequence $\left\{x_{n}\right\}$ in $X$ such that $\left\{y_{n}\right\}$ for all $n$.

## Proof:

By mathematical induction .

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## Theorem(3.5.34) Gram-Schmidt Theorem

If $\left\{x_{n}\right\}$ is an orthonormal sequence in Hilbert space $X$ and $\left\{\lambda_{n}\right\}$ is a sequence in $F$ such that $\sum_{i=1}^{\infty}\left|\lambda_{i}\right|^{2}<\infty$. Define $y_{n}=\sum_{i=1}^{n} \lambda_{i} x_{i}$, then $\left\{y_{n}\right\}$ is a Cauchy sequence, hence convergent to $y=\sum_{i=1}^{\infty} \lambda_{i} x_{i}$. Proof:

$$
y_{n}=\sum_{i=1}^{n} \lambda_{i} x_{i}, \quad y_{m}=\sum_{i=1}^{m} \lambda_{i} x_{i}
$$

If $m>n \Rightarrow m=n+k$, where $k \in \mathbb{Z}^{+}$
$y_{m}=y_{n+k}=\sum_{i=1}^{n+k} \lambda_{i} x_{i} \Rightarrow y_{m}-y_{n}=\sum_{i=n+1}^{n+k} \lambda_{i} x_{i}$
$\left\|y_{m}-y_{n}\right\|^{2}=\left\|\sum_{i=n+1}^{n+k} \lambda_{i} x_{i}\right\|^{2}=\sum_{i=n+1}^{n+k}\left|\lambda_{i}\right|^{2}\left\|x_{i}\right\|^{2}=\sum_{i=n+1}^{n+k}\left|\lambda_{i}\right|^{2} \rightarrow 0$
$\Rightarrow\left\{y_{n}\right\}$ is a Cauchy sequence, since $X$ is complete, $\Rightarrow\left\{y_{n}\right\}$ is convergent.

## Theorem(3.5.35)

Let $\left\{x_{n}\right\}$ be an orthogonal sequence in pre-Hilbert space $X$, and let $x=\sum_{i=1}^{\infty} \lambda_{i} x_{i}, y=\sum_{i=1}^{\infty} \mu_{i} x_{i}$, then
(1) $\langle x, y\rangle=\sum_{i=1}^{\infty} \lambda_{i} \overline{\mu_{i}}$
(2) $\left\langle x, x_{k}\right\rangle=\lambda_{k}$ for all $k$
(3) $\|\left. x\right|^{2}=\sum_{i=1}^{\infty}\left|\lambda_{i}\right|^{2}=\sum_{i=1}^{\infty}\left|\left\langle x, x_{i}\right\rangle\right|^{2}$

## Proof :

(1) Let $s_{n}=\sum_{i=1}^{n} \lambda_{i} x_{i}, t_{n}=\sum_{i=1}^{n} \mu_{i} x_{i} \Rightarrow s_{n} \rightarrow x, t_{n} \rightarrow y \Rightarrow\left\langle s_{n}, t_{n}\right\rangle \rightarrow\langle x, y\rangle$ $\left\langle s_{n}, t_{n}\right\rangle=\left\langle\sum_{i=1}^{n} \lambda_{i} x_{i}, \sum_{j=1}^{n} \mu_{j} x_{j}\right\rangle=\sum_{i, j} \lambda, \frac{\ell}{\mu_{j}}\left\langle x_{i}, x_{j}\right\rangle=\sum_{i, j}^{n} \lambda_{i} \overline{\mu_{j}}$
(2) Take $\mu_{j}=0, \quad \mu_{k}=1$ for all $j \neq k$, then $\left\langle x, x_{k}\right\rangle=\lambda_{k}$
(3) $\|x\|^{2}=\langle x, x\rangle=\sum_{i=1}^{\infty} \lambda_{i} \hat{\lambda}_{i}=\sum_{i=1}^{\infty}\left|\lambda_{i}\right|^{2}=\sum_{i=1}^{\infty}\left|\left\langle x, x_{i}\right\rangle\right|^{2}$

## Definition(3.5.36)

Let $A$ be a subset of pre- Hilbert $X$. We say that $A$ is a total set if $A^{\perp}=\{0\}$. In special case, a sequence $\left\{x_{n}\right\}$ is called a total sequence in $X$ if the following condition is satisfy "if $x \perp x_{n}$ for all $n$ then $x=0$ "
It is clear to show that
(1) Any pre- Hilbert $X$ is total set . because $X^{\perp}=\{0\}$
(2) The set $A=\left\{e_{1}, e_{2}, \ldots\right\}$ where $e_{n}=(0, \ldots, 0,1,0, \ldots)$ is total set in $\ell^{2}$
(3) The set $A=\left\{1, x, x^{2}, \ldots\right\}$ is total set in $C[-\pi, \pi]$.

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## Definition(3.5.37)

Let $A$ be a subset of pre- Hilbert $X$. We say that $A$ is an orthonormal basis in $X$ if it is total and orthonormal set.

## Remark

The orthonormal basis not necessary basis in $X$.

## Theorem(3.5.38)

Every Hilbert space $X$ which is not equal to zero space possesses an Orthonormal set.

## Exercise (3)

3.1 Let $A$ be a subset of normed space $X$. show that $\overline{[A]}$ is smallest closed subspace of $X$.
3.2 Let $M$ be closed subspace of normed space $X$. Show that $M \neq \mathbb{X}$ iff $\operatorname{int}(M)=\phi$
3.3 Let $A$ be the set of all converge sequences in a field $F$. Is $A$ closed subspace of $\ell^{\infty}$ ?.
3.4 Let $A_{0}$ be the set of all converge sequences to zero in a field $F$, then $A_{0}$ is complete subspace of $\ell^{\infty}$.
3.5 Let $X$ be a pre-Hilbert space . Prove or disprove
(1) $\|z-x\|^{2}+\|z-y\|^{2}=\frac{1}{2}\|x-y\|^{2}+2\left\|z-\frac{1}{2}(x+y)\right\|^{2}$ for allis $x, y, z \in X$
(2) $|\langle x, y\rangle|=\|x\| y \|$ iff $y=\lambda x$ for all $x, y \in X_{s}$ and for some $\lambda>0$
(3) $\|x+y\|=\|x\|+\|y\|$ iff $y=\lambda x$ for all $x, y \in \widehat{X}$ and for some $\lambda>0$
(4) $\|x-y\|=\|x-z\|-\|z-y\|$ iff $z=\lambda x+(1-\lambda) y$ for all $x, y \in X$ and for some $0 \leq \lambda \leq 0$
(5) $\|x-y\|=\|x\|-\|y\|$ iff $y=\lambda x$ for at ${ }^{*} x, y \in X$ and for some $\lambda>0$
(6) If $x_{n} \rightarrow 0$ and $\left\{y_{n}\right\}$ boundede, then $\left\langle x_{n}, y_{n}\right\rangle \rightarrow 0$
(7) If $\left\|x_{n}\right\| \rightarrow\|x\|$ and $\left\langle x_{n}, x\right\rangle \rightarrow\langle x, x\rangle$, then $x_{n} \rightarrow x$
(8) $\langle x, y\rangle=\operatorname{Re}(\langle x, y\rangle)+i \operatorname{Re}(\langle x, i y\rangle)$
(9) $\|x+y\|^{2}-\|x-y\|^{2}=4 \operatorname{Re}(\langle x, y\rangle)$
(10) If $\langle x, y\rangle \in \mathbb{R}$ and $\|x\| \leq 4,\|y\| \leq 2$, then $|\langle x, y\rangle| \leq 3$
(11) $x \perp y^{\prime}$ iff $\|x+\lambda y\|=\|x-\lambda y\|$ for all $\lambda \in F$
(12) $x \perp y$ iff $\|x\| \leq\|x+\lambda y\|$ for all $\lambda \in F$
(13) Let $f: X \rightarrow X$ is bounded liner and $F=\mathbb{C}$. If $\langle f(x), x\rangle=0$ for all $x \in X$, then $f=0$
(14) If $x, y$ are nonzero orthogonal vector in $X$, then $\{x, y\}$ is linear independent.
(15) If $x_{n} \rightarrow x$ and $y \perp x_{n}$ for all $n$, then $x \perp y$
3.6 Show that $M=\left\{x=\left(x_{n}\right) \in \ell^{2}: x_{2 n}=0, \quad n \in \mathbb{N}\right\}$ is closed subspace of $\ell^{2}$ and find $M^{\perp}$
3.7 Find $M^{\perp}$ if $M=\left[\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}\right] \subset \ell^{2}$, where $e_{i}=\left(\delta_{i k}\right)$

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3.8 Let $X=\mathbb{R}^{2}$. Find $M^{\perp}$ if
(1) $M=\{x\}$ (2) $M=\left\{x_{1}, x_{2}\right\}$ and $M$ is linear independent.
3.9 If $X$ is finite dimensional Hilbert space, then it has orthonormal basis.
3.10 Let $\left\{x_{n}\right\}$ be an orthonormal sequence in a pre-Hilbert space $X$ and let $x \in X$. Show that $x-y \perp M$, where $y=\sum_{k=1}^{n} \lambda_{k} x_{k}$ and $M=\left[\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}\right]$.
3.11 Show that : In a finite dimensional normed space, each closed and bounded set is compact.
3.12 Let $A$ be a subset of a Hilbert space $X$. Show that
(1) $A^{\Perp}=A^{\perp}$
(2) $A^{\perp}=[A]$ (3) $A$ is dense in $X$ iff $A^{\perp}=\{0\}$
3.13 Let $X, Y$ are Hilbert space on a field $F$. Show that $X, Y$ are Hilbert Isomorphic iff $\operatorname{dim} X=\operatorname{dim} Y$
3.14 Let $x=\left(x_{n}\right) \in \ell^{2}$. Find $\|x\|$ if (a) $x_{n}=2^{-\frac{1}{2} n} \quad$ (b) $x_{n}=\frac{1}{n}$
3.15 Show that: The colure of every compact subset of a Banach space is compact.
3.16 Show that: Suppose $X$ is a linear space that is complete with respect to norms $\|\cdot\|_{1}$ and $\|.\|_{2}$ if there exists positive real numbers $c$ such that $\|x\|_{1} \leq c\|x\|_{2}$ for all $x \in X$, then $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ are equivalent

