# 4. Continuous Linear Functions

## 4.1 Definitions and General Properties

Recall that a function $f$ from a topological space $X$ into topological space $Y$ (i.e. $f: X \to Y$) is called continuous at a point $x \in X$ if for every neighborhood $U$ of $f(x)$ in $Y$ there is a neighborhood $V$ of $x$ in $X$ such that $f(V) \subseteq U$. If $f$ is continuous at every point, it is called continuous. A function $f: X \to Y$ is continuous iff each open (rsp. closed) set $U$ in $Y$ the set $f^{-1}(U)$ is open (rsp. closed) set in $X$. A function $f: X \to Y$ from a linear space $X$ into linear space $Y$ is called a linear if $f(\sim x + \ y) = \sim f(x) + \ f(y)$ for all $x, y \in X$ and $\sim, \ \in F$.

- Linear function of a linear space $X$ into its field $F$ is called linear functional on $X$.
- Let $C(X,Y)$ denote the set of all continuous linear functions from a topological linear space $X$ into a topological linear space $Y$. Then $C(X,Y)$ is a linear space. If $Y = X$, we write $C(X)$ instead of $C(X,X)$. The space of all continuous linear functionals defined on a topological linear space $X$ is called the dual space and denoted by $X^*$, i.e. $X^* = C(X,F)$. If $X$ is finite dimensional, then $X' = X^*$

### Definition(4.1.1)

Let $(X,d)$ and $(Y,d^*)$ be metric spaces . A function $f: X \to Y$ is called an Isometry if
(1) $f$ is bijective , i.e. one –one and onto (2) $d^*(f(x), f(y)) = d(x,y)$ for all $x, y \in X$

### Theorem(4.1.2) Completion theorem

Let $(X,d)$ be an arbitrary metric space .There exists a complete metric space $(X^*, d^*)$ in which $(X,d)$ can be isometrically embedded in such a way that $X$ is dense in $X^*$, i.e. $(X,d)$ is isometric to a dense subspace of $(X^*, d^*)$.

**Note that** : All completions of metric space are isometric.

### Definition(4.1.3)

Let $X$ and $Y$ be normed spaces, An isometric isomorphism of $X$ into $Y$ is a one-one linear function $f$ of $X$ into $Y$ such that $\|f(x)\| = \|x\|$ for every $x \in X$ . Also we say that $X$ is isometrically isomorphic (or congruent) to $Y$ if there exists an isomorphism of $X$ onto $Y$ .

### Remark

Let $f$ be an isometric isomorphism of $X$ into $Y$ where $X$ and $Y$ are normed spaces. Let $x, y \in X$
Then $\|f(x) - f(y)\| = \|f(x-y)\| = \|x-y\|$
Thus $f$ preserves distances and so it is an Isometry.

### Definition(4.1.4)

Let $X$ and $Y$ be normed spaces. A topological isomorphism of $X$ into $Y$ is a one-one linear function $f$ of $X$ into $Y$ such that $f$ and $f^{-1}$ are continuous on their respective domains. Also we say that $X$ is topological isomorphic to $Y$ if there exists a topological isomorphism of $X$ onto $Y$ . In other words, $X$ and $Y$ are topologically isomorphic provided there exists a homeomorphism of $X$ onto which is also a linear function.

## Remark

Topological isomorphism space need not be isometrically isomorphic . In fact there do exists examples of pairs op spaces which are topologically isomorphic but not congruent.

## Theorem(4.1.5)

Let $X$ and $Y$ be normed spaces. Then $X$ and $Y$ are topologically isomorphic iff there exists a linear function of $X$ onto $Y$ and positive constants $\sim,\ $ such that $\sim\|x\| \le \|f(x)\| \le \ \|x\|$.

**Proof :**

Suppose $X$ and $Y$ are topologically isomorphic, then there exists a linear function $f$ of $X$ onto $Y$ such that $f$ and $f^{-1}$ are continuous. But $f$ is continuous iff there exists a positive constant such that $\|f(x)\| \le \ \|x\|$ for all $x \in X$ .

Again $f^{-1}$ is continuous iff there exists a positive constant $\sim$ such that $\sim\|x\| \le \|f(x)\|$ for all $x \in X$ .

It follows that $X$ and $Y$ are topologically isomorphic iff there exists a linear function of $X$ onto $Y$ and positive constants $\sim,\ $ such that $\sim\|x\| \le \|f(x)\| \le \ \|x\|$.

## Theorem(4.1.6)

Let $X$ and $Y$ be topological linear spaces and let $f : X \to Y$ be a linear function. If $f$ is continuous at $0$, then it is continuous

**Proof :**

Let $x \in X$ and let $U$ be a neighborhood of $f(x)$ in $Y$, then $U = f(x) + W$, where $W$ is a neighborhood of $0$ in $Y$

Since $f$ is continuous at $0$ in $X$, then there exist a neighborhood $V$ of $0$ in $X$ such that $f(V) \subset W \Rightarrow \quad x + V$ is a neighborhood of $x$ in $X$ .

To show that $f(x+V) \subset U$

Let $z \in f(x+V) \quad \Rightarrow \quad \exists\, y \in x+V$ such that $z = f(y)$

Since $y \in x+V \quad \Rightarrow \quad y - x \in V \quad \Rightarrow \quad f(y-x) \in f(V)$

$f(y) - f(x) \in f(V) \quad \Rightarrow \quad z - f(x) \in f(V) \quad \Rightarrow \quad z \in f(x) + f(V) \quad \Rightarrow \quad z \in U$

$\Rightarrow f$ is continuous at $x$, then $f$ is continuous.

## Theorem(4.1.7)

Let $X$ and $Y$ be normed spaces and let $f : X \to Y$ be a linear function. Then $f$ is continuous either at every point of $X$ or at no point of $X$ .

**Proof :**

Let $x_1$ and $x_2$ be any two points of $X$ and suppose $f$ is continuous at $x_1$.Then to each $£ > 0$ there exists $¢ > 0$ such that

$$\|x - x_1\| < ¢ \quad \Rightarrow \quad \|f(x) - f(x_1)\| < £$$

Now

$\|x - x_2\| < ¢ \quad \Rightarrow \quad \|(x + x_1 - x_2) - x_1\| < ¢ \quad \Rightarrow \quad \|f(x + x_1 - x_2) - f(x_1)\| < £$

$\Rightarrow \quad \|f(x) + f(x_1) - f(x_2) - f(x_1)\| < £ \quad \Rightarrow \quad \|f(x) - f(x_2)\| < £$

$\Rightarrow$ $f$ is continuous at $x_2$, then $f$ is continuous.

## Lemma (4.1.8)

Let $X$ and $Y$ be Banach spaces and $f : X \to Y$ a continuous, linear and onto function. Then the image of each open ball centered on the origin in $X$ contains an open ball centered on the origin in $Y$ .

Proof : H.W

## Theorem(4.1.9) The open mapping theorem

Let $X$ and $Y$ be Banach spaces . If $f : X \to Y$ is a continuous, linear and onto function, then $f$ is open

Proof :

Let $G$ be open set in $X$ . We want to show that $f(G)$ is open in $Y$

Let $y \in f(G)$, then $y = f(x)$ for some $x \in G$

Since $G$ is open set in $X$ ,there exist $r > 0$ such that $\ _r(x) \subseteq G \quad \Rightarrow \quad f(\ _r(x)) \subseteq f(G)$

Since $\ _r(x) = x + \ _r(0) \quad \Rightarrow \quad x + \ _r(0) \subseteq G$

By our lemma, there exists an open sphere $\ '_r(0)$ in $Y$ centered at origin such that

$\ '_r(0)_r \subseteq f(\ _r(0))$

$\Rightarrow \quad y + \ '_r(0) \subseteq y + f(\ _r(0)) = f(x) + f(\ _r(0)) = f(x + \ _r(0)) = f(\ _r(x)) \subseteq f(G)$

Since $y + \ '_r(0) = \ '_r(y) \quad \Rightarrow \quad \ '_r(y) \subseteq f(G) \quad \Rightarrow \quad f(G)$ is open , thus $f$ is an open .

The following special case of the above theorem is very important.

## Theorem(4.1.10)

Let $X$ and $Y$ be Banach spaces . If $f : X \to Y$ is a bijection continuous linear function, then $f$ is homeomorphism.

Proof :

Since $f$ is bijection continuous function, we need only prove that $f$ is an open function.

## Definition(4.1.11)

Let $X$ and $Y$ be any non-empty sets and let $f : X \to Y$ be a function. The set

$$\{(x,y) \in X \times Y \ \ y = f(x)\} = \{(x, f(x)) : x \in X, f(x) \in Y\}$$

is called the graph of $f$ . We shall denote the graph of $f$ by $f_G$ .i.e.

$$f_G = \{(x,y) \in X \times Y \ \ y = f(x)\} = \{(x, f(x)) : x \in X, f(x) \in Y\} .$$

In the case $X$ and $Y$ are normed spaces .Then $X \times Y$ is a normed spaces

We will now generalize the above notion of graph

**Closed Linear Function**

## Definition(4.1.12)

Let $X$ and $Y$ be normed Spaces and let $D$ be a subspace of $X$ .The linear function $f:D\to Y$ is called closed if for every sequence $\{x_n\}$ in $D$ such that $x_n\to x\in X$ and $f(x_n)\to y$ , then $x\in D$ and $y=f(x)$.

## Theorem(4.1.13)

Let $X$ and $Y$ be normed Spaces and let $D$ be a subspace of $X$ . The linear function $f:D\to Y$ is closed iff its graph $f_G$ is closed subspace.

**Proof:**

Suppose that $f:D\to Y$ is closed .to show that $f_G$ is closed subspace.

Let $(x,y)$ be any limit point of $f_G$ ,i.e. $(x,y)\in\overline{f_G}$ , then there exists a sequence of points in $f_G$ , $(x_n,f(x_n))$ where, $x_n\in D$ such that $(x_n,f(x_n))\to(x,y)$

$\Rightarrow\ (x_n,f(x_n))-(x,y)\to 0\ \Rightarrow\ \|(x_n,f(x_n))-(x,y)\|\to 0\ \Rightarrow\ \|(x_n-x,f(x_n)-y)\|\to 0$

$\|x_n-x\|+\|f(x_n)-y\|\to 0\ \Rightarrow\ \|x_n-x\|\to 0$ and $\|f(x_n)-y\|\to 0\ \Rightarrow\ x_n\to x$ and $f(x_n)\to y$

Since $f:D\to Y$ is closed, then $x\in D$ and $y=f(x)\Rightarrow\ (x,y)\in f_G\ \Rightarrow\ f_G$ is closed .

Conversely , let the graph $f_G$ is closed. To show that The linear function $f:D\to Y$ is closed.

Let $\{x_n\}$ be a sequence in $D$ such that $x_n\to x\in X$ and $f(x_n)\to y\ \Rightarrow\ (x_n,f(x_n))\to(x,y)$

$\Rightarrow\ (x,y)\in\overline{f_G}$ , since $f_G$ is closed $\Rightarrow\ \overline{f_G}=f_G\ \Rightarrow\ (x,y)\in f_G\ \Rightarrow\ x\in D$ and $y=f(x)$.

Therefore The linear function $f:D\to Y$ is closed.

## Theorem(4.1.14) Closed Graph Theorem

Let $X$ and $Y$ be Banach spaces . If $f:X\to Y$ is a linear function, then $f$ is continuous iff its graph is closed

**Proof :**

Suppose that $f$ is continuous .To show that $f_G=\{(x,f(x)):x\in X,f(x)\in Y\}$ is closed

Let $(x,y)$ be any limit point of $f_G$ ,i.e. $(x,y)\in\overline{f_G}$ , then there exists a sequence $(x_n,f(x_n))$ in $f_G$ such that $(x_n,f(x_n))\to(x,y)$

$\Rightarrow\ (x_n,f(x_n))-(x,y)\to 0\ \Rightarrow\ \|(x_n,f(x_n))-(x,y)\|\to 0\ \Rightarrow\ \|(x_n-x,f(x_n)-y)\|\to 0$

$\|x_n-x\|+\|f(x_n)-y\|\to 0\ \Rightarrow\ \|x_n-x\|\to 0$ and $\|f(x_n)-y\|\to 0\ \Rightarrow\ x_n\to x$ and $f(x_n)\to y$

Since $f:X\to Y$ is continuous and $x_n\to x$ , then $f(x_n)\to f(x)$

Since $f(x_n)\to y$ , then $y=f(x)\Rightarrow\ (x,y)=(x,f(x))\in f_G\ \Rightarrow\ f_G$ is closed.

Conversely , let $f_G$ be closed. To show that $f$ is continuous(H.w)

## 4.2 Boundedeness

### Definition(4.2.1)

Let $X$ and $Y$ be topological linear spaces and let $f : X \to Y$ be a linear function. We say that $f$ is bounded if $f(A)$ is bounded subset of $Y$ for every bounded subset $A$ of $X$.

### Theorem(4.2.2)

Let $X$ be a topological linear spaces over $F$ .Then $f : X \to F$ is continuous at $0$ in $X$ . If for every $r > 0$, there exists a neighborhood $V$ at $0$ in $X$ such that $|f(x)| < r$ for all $x \in V$.

### Theorem(4.2.3)

Let $X$ be a Hausdorff topological linear spaces over $F$ and $f \in X'$ . Assume $f(x) \neq 0$ for some $x \in X$ . Then the following statements are equivalent

(1) $f$ is continuous

(2) $\ker(f)$ is closed

(3) $\ker(f)$ is not dense in $X$ .

(4) $f$ is bounded in some neighborhood $V$ of $0$ in $X$ .

**Proof** :

(1) $\Rightarrow$ (2)

Since $\{0\}$ is closed in $F$ and $f : X \to F$ is continuous, then $f^{-1}(\{0\})$ is closed in $X$

$\Rightarrow \ker(f) = f^{-1}(\{0\})$ is closed in $X$

(2) $\Rightarrow$ (3)

Since $\ker(f)$ is closed in $X \Rightarrow \overline{\ker(f)} = \ker(f)$

Since $f(x) \neq 0$ for some $x \in X$ , then $x \notin \ker(f) \Rightarrow \ker(f) \neq X \Rightarrow \ker(f)$ is not dense in $X$

(3) $\Rightarrow$ (4)

Let $A = X \,|\, \overline{\ker(f)}$. Since $\overline{\ker(f)} \neq X \Rightarrow A \neq \emptyset$

Since $\overline{\ker(f)}$ is closed in $X$ , then $A$ is open set in $X \Rightarrow \operatorname{int}(A) = A \neq \emptyset$ , then there is $x \in \operatorname{int}(A) \Rightarrow x + V \subseteq A$ for some balanced neighborhood $V$ of is $0$ in $X$ , since $f$ is a linear, then $f(V)$ is balanced set in $F \Rightarrow$ either $f(V)$ is bounded or $f(V) = F$ .

If $f(V) = F$, then there is $y \in V$ such that $f(y) = -f(x) \Rightarrow f(x + y) = 0 \Rightarrow x + y \in \ker(f)$

$\Rightarrow (x + V) \cap \ker(f) \neq \emptyset$ . This contradiction, so that $f(V)$ is bounded

(4) $\Rightarrow$ (1)

Since $f$ is bounded in some neighborhood $V$ of $0$ in $X$ .

$\exists M > 0$, such that $|f(x)| < M$ for all $x \in V$

Let $r > 0$, take $W = \frac{r}{M} V \Rightarrow W$ is a neighborhood of $0$ in $X$

Let $y \in W \;\Rightarrow\; y = \dfrac{r}{M}x$ where $x \in V$

$$|f(y)| = \left|f\left(\frac{r}{M}x\right)\right| = \left|\frac{r}{M}f(x)\right| = \frac{r}{M}|f(x)| < \frac{r}{M}.M = r$$

$|f(y)| < r$ for all $y \in W \;\Rightarrow\; f$ is continuous at $0$ in $X$, then $f$ is continuous.

## Example(4.2.4)

Let $X$ and $Y$ be normed spaces and let $f : X \to Y$ be a linear function. If $f$ is continuous, then $\ker(f)$ is closed, but the converse is not true.

**Ans :**

Since $\{0\}$ is closed in $Y$ and $f : X \to Y$ is continuous, then $f^{-1}(\{0\})$ is closed in $X$

$\Rightarrow \ker(f) = f^{-1}(\{0\})$ is closed in $X$

The counter example

Let $X = C^1[0,1]$ and $Y = C[0,1]$ with the same norm $\|\Psi\| = \sup\{\Psi(x) : 0 \le x \le 1\}$

Define $f : X \to Y$ by $f(\Psi) = \dfrac{d\Psi}{dx}$

Then $\ker(f)$ is the set of all constant functions, then $\ker(f)$ is closed, but $f$ is not continuous because, if $\Psi_n(x) = x^n$ for all $x \in [0,1]$, then $\|\Psi_n\| = 1$, but $\|f(\Psi_n)\| = \|nx^{n-1}\| = n$ for all $n = 0,1,2,\cdots$

## Theorem(4.2.5)

Let $X$ and $Y$ be topological linear spaces and let $f : X \to Y$ be a linear function. Among the following four properties of $f$, the implications $(1) \Rightarrow (2) \Rightarrow (3)$ hold, If $X$ is metrizable, then also $(3) \Rightarrow (4) \Rightarrow (1)$

So that all four properties are equivalent.

(1) $f$ is continuous

(2) $f$ is bounded

(3) If $x_n \to 0$, then $\{f(x_n)\}$ is bounded

(4) If $x_n \to 0$, then $f(x_n) \to 0$

**Proof :**

$(1) \Rightarrow (2)$

Let $A$ be a bounded set in $X$ and let $W$ be a neighborhood of $0$ in $Y$ (since $(f(0) = 0)$)

Since $f$ is continuous, then $f$ is continuous at $0$, there is a neighborhood $V$ of $0$ in $X$ such that $f(V) \subset W$.

Since $A$ is bounded, then there is $\alpha > 0$ such that $A \subset \alpha V$

$\Rightarrow f(A) \subset f(\alpha V) = \alpha f(V) \subset \alpha W \;\Rightarrow\; f(A)$ is bounded in $Y \;\Rightarrow\; f$ is bounded.

(2) $\Rightarrow$ (3)

Since $x_n \to 0 \Rightarrow \{x_n\}$ is bounded (because every converge sequence is bounded)
Since $f$ is bounded $\Rightarrow \{f(x_n)\}$ is bounded.

(3) $\Rightarrow$ (4)

Since $X$ is metrizable and $x_n \to 0$, by theorem () there are positive scalar $\sim_n$ such that $\sim_n \to \infty$ and $\sim_n x_n \to 0$ , we have $\{f(\sim_n x_n)\}$ is bounded

Since $\sim_n \to \infty \Rightarrow \text{ª}_n = \frac{1}{\sim_n} \to 0$, then $f(x_n) = \text{ª}_n f(\sim_n x_n) \to 0$ as $n \to \infty$

(4) $\Rightarrow$ (1)

Assume $f$ is not continuous
There exists a neighborhood $W$ of $0$ in $Y$ such that $f^{-1}(W)$ contains no neighborhood of $0$ in $X$.
If $X$ has a countable local base, there is a sequence $\{x_n\}$ in $X$, so that $x_n \to 0$ but $f(x_n) \notin W$ (i.e. $f(x_n) \not\to 0$ ). Thus (4) fails, so that $f$ is continuous.

## Remark

Recall that, a subset $A$ of a normed space $X$ is bounded iff there is $k > 0$ such that $\|x\| \le k$ for all $x \in A$.

## Theorem(4.2.6)

Let $X$ and $Y$ be normed spaces and let $f : X \to Y$ be a linear function. Then $f$ is bounded iff there is $k > 0$ such that $\|f(x)\| \le k\|x\|$ for all $x \in X$.

**Proof :**

Suppose that $f$ is bounded, since $A = \{x \in X : \|x\| \le 1\}$ is bounded in $X \Rightarrow A$ is bounded in $X$.
Since , then $f$ is bounded, then $f(A)$ is bounded in $Y$
There is $k > 0$ such that $\|f(x)\| \le k$ for all $x \in A$.
Let $x \in X$
If $x \ne 0$, put $y = \frac{x}{\|x\|} \Rightarrow \|y\| = 1 \Rightarrow y \in A$

$$\|f(y)\| \le k \Rightarrow \left\| f\left(\frac{x}{\|x\|}\right) \right\| \le k \Rightarrow \frac{1}{\|x\|}\|f(x)\| \le k \Rightarrow \|f(x)\| \le k\|x\|$$

either if $x = 0$,then $f(x) = f(0) = 0$, so $\|f(x)\| \le k\|x\|$ for all $x \in X$
Conversely

There is $k > 0$ such that $\|f(x)\| \le k\|x\|$ for all $x \in X$.

Let $A$ be a bounded set in $X$, then there is $k_1 > 0$ such that $\|x\| \le k_1$ for all $x \in A$
$\Rightarrow k\|x\| \le kk_1 = k_2$ for all $x \in A$

Since $\|f(x)\| \le k\|x\|$ for all $x \in X$, then $\|f(x)\| \le k\|x\|$ for all $x \in A$

$\Rightarrow$ $\|f(x)\| \le k_2$ for all $x \in A \Rightarrow$ $f(A)$ is bounded in $Y$, then $f$ is bounded.

## Theorem(4.2.7)

Let $X$ and $Y$ be normed spaces and let $f : X \to Y$ be a linear function. Then $f$ is bounded iff it is continuous.

**Proof :**

Let $f$ be a bounded, then there is $k > 0$ such that $\|f(x)\| \le k\|x\|$ for all $x \in X$.

Let $x_0 \in X$. For any $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{k}$

$\|x - x_0\| < \delta$, we have $\|f(x) - f(x_0)\| = \|f(x - x_0)\| \le k\|x - x_0\|$ $\Rightarrow$ $\|f(x) - f(x_0)\| < k.\delta = \varepsilon$, hence $f$ is continuous at $x_0$. Since $x_0$ is arbitrary $\Rightarrow$ $f$ is continuous

**Conversely :** Assume that $f$ is unbounded

For each positive integer $n$, we can find a vector $x_n$ such that $\|f(x_n)\| > n\|x_n\|$

$$\Rightarrow \quad \frac{1}{n\|x_n\|}\|f(x_n)\| > 1 \quad \Rightarrow \quad \left\| f\left(\frac{x_n}{n\|x_n\|}\right)\right\| > 1$$

Put $y_n = \dfrac{x_n}{n\|x_n\|}$ $\Rightarrow$ $\|y_n\| = \dfrac{1}{n}$ $\Rightarrow$ $\|y_n\| \to 0$ as $n \to \infty$ $\Rightarrow$ $y_n \to 0$ as $n \to \infty$

Since $f$ is continuous, then $f(y_n) \to f(0) = 0$

$\Rightarrow$ $\|f(y_n)\| \to 0$. This contradiction, because $\|f(y_n)\| > 1$, then $f$ is bounded.

## Theorem(4.2.8)

Let $X$ and $Y$ be normed spaces and let $f : X \to Y$ be a linear function. If $X$ is finite dimensional, then $f$ is bounded (continuous).

**Proof :**

Let $\dim X = n$, and let $\{x_1, \dots, x_n\}$ be a basis for $X$, then every $x \in X$ has a unique representation,

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_i \in F, \qquad i = 1, 2, \cdots, n$$

$$f(x) = \sum_{i=1}^{n} \alpha_i f(x_i) \qquad \Rightarrow \qquad \|f(x)\| = \left\| \sum_{i=1}^{n} \alpha_i f(x_i) \right\| \le \sum_{i=1}^{n} |\alpha_i| \, \|f(x_i)\|$$

Put $k = \max\{\|f(x_1)\|, \dots, \|f(x_n)\|\}$, then $\|f(x)\| \le k\sum_{i=1}^{n} |\alpha_i| \qquad \cdots \quad (1)$

Since the set $\{x_1,...,x_n\}$ is linear independent, by lemma of combination, there is $c>0$ such that $\|x\|=\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|\geq c\sum_{i=1}^{n}|\alpha_i| \quad\Rightarrow\quad \sum_{i=1}^{n}|\alpha_i|\leq\frac{1}{C}\|x\| \quad\cdots\quad (2)$

From (1), (2), we have $\|f(x)\|\leq\frac{k}{C}\|x\|$, so that $f$ is bounded.

## 4.3 Spaces of Bounded Linear Functions

Let $B(X,Y)$ denote the set of all bounded linear functions from a normed space $X$ into a normed space $Y$.

### Definition(4.3.1)

Let $X$ and $Y$ be normed spaces over $F$ and let $f:X\to Y$ be a linear function. We define the norm of $f$ by $\|f\|=\sup\{\|f(x)\|:x\in X,\ \ \|x\|\leq 1\}$

### Theorem(4.3.2)

Let $X$ and $Y$ be normed spaces over $F$ and let $f:X\to Y$ be a linear function. If

$a=\sup\{\|f(x)\|:x\in X,\ \ \|x\|=1\}$, $b=\sup\left\{\frac{\|f(x)\|}{\|x\|}:x\in X,\ \ x\neq 0\right\}$,

$c=\inf\{\alpha>0:\|f(x)\|\leq\alpha\|x\|\quad\forall x\in X\}$. Then $\|f\|=a=b=c$ and $\|f(x)\|\leq\|f\|\|x\|$ for all $x\in X$

**Proof :**

By definition of norm $\|f\|=\sup\{\|f(x)\|:x\in X,\ \ \|x\|\leq 1\}$, by definition of $c$, we have $\|f(x)\|\leq c\|x\|$ for $x\in X$

if $\|x\|\leq 1 \ \Rightarrow\ c\|x\|\leq c$ for $x\in X \ \Rightarrow\ \|f(x)\|\leq c$ for $x\in X$ and $\|x\|\leq 1$

$\Rightarrow\ \sup\{\|f(x)\|:x\in X,\ \|x\|\leq 1\}\leq c \quad\Rightarrow\quad \|f\|\leq c \qquad\cdots(1)$

Also by definition of $b$, we have $\|f(x)\|\leq b\|x\|$ for all $x\neq 0$

Since $c=\inf\{\alpha>0:\|f(x)\|\leq\alpha\|x\|\ \ \forall x\in X\}$, we have $c\leq b \qquad\cdots(2)$

let $x\in X,\ \ x\neq 0$

$$\frac{\|f(x)\|}{\|x\|}=\frac{1}{\|x\|}\|f(x)\|=\left\|f\left(\frac{x}{\|x\|}\right)\right\|$$

Put $y=\frac{x}{\|x\|}\ \Rightarrow\ \|y\|=1\ \Rightarrow\ y\in X \quad\Rightarrow\quad b\leq a \qquad\cdots(3)$

It is clear to show that $a\leq\|f\|$, so that $\|f\|=a=b=c$.

Finally definition of $b$, shows that

$b\geq\frac{\|f(x)\|}{\|x\|} \quad\Rightarrow\quad \|f(x)\|\leq b\|x\|$ for all $x\in X$

But $\|f\|=b\ \Rightarrow\ \|f(x)\|\leq\|f\|\|x\|$ for all $x\in X$

## Theorem(4.3.3)

Let $X$ and $Y$ be normed spaces over $F$ .Then $B(X,Y)$ is normed space with respect to the norm defined by $\|f\| = \sup\{\|f(x)\| : x \in X, \|x\| \le 1\}$ for all $f \in B(X,Y)$ .

**Proof :**

(1) Since $\|f(x)\| \ge 0$ for all $x \in X$ , then $\|f\| \ge 0$ for all $x \in X$

(2) $\|f\| = 0 \Leftrightarrow \sup\{\|f(x)\| : x \in X , \ \|x\| \le 1\} = 0$

$$\Leftrightarrow \sup\{\frac{\|f(x)\|}{\|x\|} : x \in X , \quad x \ne 0\} = 0 \quad \Leftrightarrow \frac{\|f(x)\|}{\|x\|} = 0 : x \in X , \quad x \ne 0 \quad \Leftrightarrow \|f(x)\| = 0 : x \in X$$

$$\Leftrightarrow f(x) = 0 : x \in X \Leftrightarrow f = 0$$

(3) Let $f \in B(X,Y)$ and $\alpha \in F$

$$\|\alpha f\| = \sup\{\|(\alpha f)(x)\| : x \in X , \quad \|x\| \le 1\} = \sup\{|\alpha| \|f(x)\| : x \in X , \quad \|x\| \le 1\}$$

$$= |\alpha| \sup\{\|f(x)\| : x \in X , \quad \|x\| \le 1\} \ |\alpha| \|f\|$$

(4) Let $f, g \in B(X,Y)$

$$\|f + g\| = \sup\{\|(f + g)(x)\| : x \in X , \quad \|x\| \le 1\}$$

$$= \sup\{\|f(x) + g(x)\| : x \in X , \quad \|x\| \le 1\}$$

$$\le \sup\{\|f(x)\| + \|g(x)\| : x \in X , \quad \|x\| \le 1\}$$

$$\le \sup\{\|f(x)\| : x \in X , \quad \|x\| \le 1\} + \sup\{\|g(x)\| : x \in X , \quad \|x\| \le 1\}$$

$$= \|f\| + \|g\|$$

$\Rightarrow \quad B(X,Y)$ normed space

## Theorem(4.3.4)

Let $X$ and $Y$ be normed spaces over $F$ .If $Y$ is Banach space, then $B(X,Y)$ is also Banach space.

**Proof :**

$B(X,Y)$ is a normed space (by above theorem)

Let $\{f_n\}$ be a Cauchy sequence in $B(X,Y)$, then $\|f_n - f_m\| \to 0$ as $n,m \to \infty$

For all $x \in X$ , then $\|f_n(x) - f_m(x)\| = \|(f_n - f_m)(x)\| \le \|f_n - f_m\| \ \|(x)\|$

$\Rightarrow \quad \|f_n(x) - f_m(x)\| \to 0$ as $n,m \to \infty \ \Rightarrow \quad \{f_n(x)\}$ is Cauchy sequence in $Y$ for all $x \in X$

Since $Y$ is complete, then $f(x) \in Y$ such that $f_n(x) \to f(x)$, then $f \in B(X,Y)$ why? , so that $\{f_n\}$ converge , then $B(X,Y)$ is Banach

## Corollary(4.3.5)

If $X$ is a normed space over $F$ , then $X^*$ is a Banach space.

## Example (4.3.6)

The dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$ , i.e. $(\mathbb{R}^n)^* \approx \mathbb{R}^n$

**Ans :**

Since $\mathbb{R}^n$ is finite dimensional, then $(\mathbb{R}^n)^* = (\mathbb{R}^n)'$
let $\{x_1, \cdots, x_n\}$ be a basis for $\mathbb{R}^n$, then every $x \in \mathbb{R}^n$ has a unique representation,

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_i \in \mathbb{R}, \qquad i = 1, 2, \cdots, n$$

$$f(x) = f(\sum_{i=1}^{n} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i f(x_i) = \sum_{i=1}^{n} \alpha_i y_i, \qquad y_i = f(x_i), \qquad i = 1, \cdots, n$$

By using the Cauchy- Schwarz inequality, we have

$$|f(x)| \le \sum_{i=1}^{n} |\alpha_i y_i| \le ((\sum_{i=1}^{n} |\alpha_i|^2)^{\frac{1}{2}})((\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}}) \Rightarrow |f(x)| \le \|x\| (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$$

$$\|f\| = \sup\{|f(x)| : x \in R^n, \quad \|x\| = 1\} \quad \Rightarrow \quad \|f\| \le (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$$

This proves that the norm of $f$ is the norm of $\mathbb{R}^n$, i.e. $\|f\| = (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$

$\Rightarrow \|f\| = \|y\|$, where $Y = (y_1, ..., y_n) \in R^n$ .Hence the function $\cdot : (R^n)' \to R^n$ defined by $\cdot(f) = y = (y_1, ..., y_n)$ where $y_i = f(x_i)$ it is linear and bijective, it is an isomorphism. So that $(\mathbb{R}^n)^* \approx \mathbb{R}^n$.

## Example (4.3.7)

The dual space of $\ell^1$ is $\ell^\infty$

**Ans :**

Let $\{e_k\}$ be a natural basis for $\ell^1$ where $e_k = (\delta_{ki})$, i.e.

$e_1 = (1,0,0,\cdots), e_2 = (0,1,0,\cdots), e_3 = (0,0,1,\cdots), \cdots$

Then every $x \in \ell^1$ has a unique representation, $x = \sum_{k=1}^{\infty} \alpha_k e_k$ where $\alpha_k \in F$

We consider any $f \in (\ell^1)^* \Rightarrow f$ is bounded linear functional on $\ell^1$

$$f(x) = f(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f(e_k) = \sum_{k=1}^{\infty} \alpha_k y_k, \quad , \quad y_k = f(e_k)$$

Where $y_k = f(e_k)$ has a unique representation by $f$ .Also $\|e_k\| = 1$ and

$$|y_k| = |f(e_k)| \le \|f\| \|e_k\| = \|f\| \quad \Rightarrow \quad \sup_k |y_k| \le \|f\| \quad \Rightarrow \quad y = (y_k) \in \ell^\infty$$

On the other hand, let $Z = (Z_k) \in \ell^\infty$, define $g : \ell^1 \to F$ by $g(x) = \sum_{k=1}^{\infty} \alpha_k z_k$ where

$x = (x_k) \in \ell^1 \Rightarrow g$ is bounded linear

$$|g(x)| \le \sum_{k=1}^{\infty} |x_k z_k| \le \sup_j |z_k| \sum_{k=1}^{\infty} |x_k| = \|x\| \sup_j |z_k| \quad \Rightarrow \quad g \in (\ell^1)^*$$

We finally to show that $\|f\| = \sup_j |y_j|$

$$|f(x)| = \left|\sum_{k=1}^{\infty} \alpha_k y_k\right| \le \sup_j |y_j| \left|\sum_{k=1}^{\infty} |\alpha_k|\right| = \|x\| \sup_j |y_j| \quad \Rightarrow \quad \|f\| \le \sup_j |y_j| \text{ , so that } \|f\| = \sup_j |y_j|$$

Hence the function $\Psi = (\ell^1)^* \to \ell^\infty$ defined by $\Psi(f) = (y_j)$ where $y_j = f(e_j)$ it is linear and bijective, it is an isomorphism.

## Example (4.3.8)

The dual space of , $\ell^p$, $1 < p < \infty$ is $\ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$

**Ans :**

Let $\{e_k\}$ be a natural basis for $\ell^p$ where $e_k = (\delta_{ki})$, i.e.

$$e_1 = (1,0,0,\cdots), e_2 = (0,1,0,\cdots), e_3 = (0,0,1,\cdots), \cdots$$

Then every $x \in \ell^p$ has a unique representation, $x = \sum_{k=1}^{\infty} \alpha_k e_k$ where $\alpha_k \in F$

We consider any $f \in (\ell^p)^* \;\Rightarrow\; f$ is bounded linear functional on $\ell^p$

$$f(x) = f\Big(\sum_{k=1}^{\infty} \alpha_k e_k\Big) = \sum_{k=1}^{\infty} \alpha_k f(e_k) = \sum_{k=1}^{\infty} \alpha_k y_k \,, \quad , \quad y_k = f(e_k)$$

Let $q \in \mathbb{R}$, where $\frac{1}{p} + \frac{1}{q} = 1$

Put $x_n = (\alpha_{k_n})$, where $\alpha_{kn} = \begin{cases} \dfrac{|y_k|^q}{y_k}, & k \le n \,, \quad y_n \ne 0 \\ 0, & o.w \end{cases}$

$$f(x_n) = \sum_{k=1}^{\infty} \alpha_n y_k = \sum_{k=1}^{n} |y_k|^q$$

$$f(x_n) \le \|f\| \|x_n\| = \|f\| \Big(\sum |\alpha_{kn}|^p\Big)^{\frac{1}{p}} = \|f\| \Big(\sum |y_k|^{(q-1)p}\Big)^{\frac{1}{p}} = \|f\| \Big(\sum |y_k|^q\Big)^{\frac{1}{p}}$$

$$f(x_n) = \sum |y_k|^q \le \|f\| \Big(\sum |y_k|^q\Big)^{\frac{1}{p}} \quad \Rightarrow \quad \Big(\sum_{k=1}^{n} |y_k|^q\Big)^{1-\frac{1}{p}} = \Big(\sum_{k=1}^{n} |y_k|^q\Big)^{\frac{1}{q}} \le \|f\|$$

Since $n$ is arbitrary, letting $n \to \infty$, we obtain $\Big(\sum_{k=1}^{\infty} |y_k|^q\Big)^{\frac{1}{q}} \le \|f\| \quad \Rightarrow \quad (y_k) \in \ell^q$

On the other hand, let $Z = (Z_k) \in \ell^\infty$, define $g : \ell^1 \to F$ by $g(x) = \sum_{k=1}^{\infty} \alpha_k z_k$ where $x = (\alpha_k) \in \ell^p$

$\Rightarrow \quad g$ is bounded linear

$$|f(x)| = \left|\sum \alpha_k y_k\right| \le \Big(\sum_{k=1}^{\infty} |\alpha_k|^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{\infty} |y_k|^q\Big)^{\frac{1}{q}} = \|x\| \Big(\sum_{k=1}^{\infty} |y_k|^q\Big)^{\frac{1}{q}}$$

$\|f\| \leq (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}$ so that $\|f\| \leq (\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}$

Hence the function $\Psi : (\ell^p)^* \to \ell^q$ defined by $\Psi(f) = (y_k)$ where $y_k = f(e_k)$ it is linear and bijective, it is an isomorphism.

## Definition(4.3.9)

Let $X$ be a normed Space over a filed $F$ .We define $X^{**}$ as :

$$X^{**} = (X^*)^* = \{G : X^* \to F, G \text{ is bounded linear functional } \}$$

$X^{**}$ is called the second dual space.

## Theorem(4.3.10)

Let $X$ be a normed Space over a filed $F$ .

(1) If $x \in X$ and $T_x : X^* \to F$ defined as $T_x(f) = f(x)$ for all $f \in X^*$, then $T_x \in X^{**}$ and $\|T_x\| = \|x\|$

(2) If $\cdot : X \to X''$ defined as $\cdot(x) = T_x$ for all $x \in X$, then $\cdot$ is one-one linear function.

**Proof :**

(1) $T_x$ is linear (see theorem8.1)

$$\|T_x\| = \sup\left\{\frac{|T_x(f)|}{\|f\|} : f \in X^*, \quad f \neq 0\right\} = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, \quad f \neq 0\right\} = \|x\|$$

(2) (see theorem1.3.4)

## Definition(4.3.11)

Let $X$ be a normed Space over a filed $F$. We say that $X$ is Reflexive space if $\cdot$ is onto, where $\cdot$ is canonical function defined in theorem (4.3.10).

It is clear to show that

(1) If $X$ is reflexive space, then $X \simeq X^{**}$ (2) Every finite dimensional normed space is reflexive.

## Theorem(4.3.12)

Let $X$ be a normed space. If $X$ is reflexive, then $X$ is complete, and hence it is Banach space.

**Proof :**

Since $X^*$ is normed space $\Rightarrow$ $X^{**}$ is complete space

Since $X$ is reflexive $\Rightarrow$ $X \simeq X^{**} \Rightarrow$ $X$ is complete space.

# 4.4 Separable Spaces

Recall that a subset $A$ of a topological space $X$ is said to be dense in $X$ if $\overline{A} = X$ . and a topological space $X$ is said separable if it has a countable subset which is dense in $X$ .

## Examples(4.4.1)

(1)The space $\mathbb{R}$ is separable, because the set $\mathbb{Q}$ of rational numbers is countable and is dense in $\mathbb{R}$.

(2)The space $\mathbb{C}$ is separable, because a countable subset of $\mathbb{C}$ is the set of all complex numbers whose real and imaginary parts both rational .

(3) A discrete metric space is separable iff $X$ is separable .

## Examples(4.4.2)

The space $\ell^{\infty}$ is not separable

**Ans :**

Let $A$ be a countable set in $\ell^{\infty} \Rightarrow \quad A=\{x_1, x_2, \cdots\}$ where $x_n=(x_{1n}, x_{2n}, \cdots) \in \ell^{\infty}$

Let $y=(y_k) \in \ell^{\infty}$ where $y_k=\begin{cases} x_{kk}+1 & , \quad |x_{kk}| \leq 1 \\ \\ 0 & , \quad |x_{kk}|>1 \end{cases}$

The component $k$ of $y-x_k$ is $y_k - x_{kk}$ , $|y_k - x_{kk}| \geq 1$

$\Rightarrow \quad \|y-x_k\| \geq 1 \quad \Rightarrow \quad y \notin \overline{A} \quad \Rightarrow \quad \overline{A} \neq \ell^{\infty}$ for all countable subset $A$ of $\ell^{\infty}$ . $\Rightarrow \quad \ell^{\infty}$ is not separable.

## Remark

An element $x=(x_n) \in \ell^{p}$ is called rational if

(1) $x_n \in \mathbb{Q}$ for all $n$ , if $F=\mathbb{R}$

(2) both the real and imaginary parts are rationales, if $F=\mathbb{C}$

## Example(4.4.3)

The space $\ell^{p}$ with $1 \leq p<\infty$ is separable

**Ans :**

Let $A=\{x=(x_1, x_2, \cdots, x_n, 0,0, \cdots) \in \ell^{p} : x$ is rational $\} \Rightarrow \quad A$ is countable .

we shall to prove $\ell^{p} \subset \overline{A}$ (since $\ell^{p} \subset \overline{A}$)

Let $y=(y_i) \in \ell^{p} \Rightarrow \quad \sum_{i=1}^{\infty}|y_i|^{p}<\infty$

Then for every $\varepsilon>0$, there is an $m \in \mathbb{Z}^{+}$ (depending on $\varepsilon$ ) such that $\sum_{i=m+1}|y_i|^{p}<\frac{\varepsilon^{p}}{2}$

Hence we can find a $x=(x_1, x_2, \cdots, x_n, 0,0, \cdots) \in A$ satisfying $|x_i-y_i|^{p}<\frac{\varepsilon^{p}}{2m}$ , for all $i=1,2, \cdots, m$

$$\|x-y\|^{p}=\sum_{i=1}^{\infty}|x_i-y_i|^{p}=\sum_{i=1}^{m}|x_i-y_i|^{p}+\sum_{i=m+1}^{\infty}|x_i|^{p}<m \cdot \frac{\varepsilon^{p}}{2m}+\frac{\varepsilon^{p}}{2}=\varepsilon^{p}$$

$\Rightarrow \quad \|x-y\|<\varepsilon \quad \Rightarrow \quad y \in \overline{A} \quad \Rightarrow \quad \overline{A}=\ell^{p} \Rightarrow \quad \ell^{p}$ is separable .

## Theorem(4.4.4)

A normed space $X$ over $F$ is separable if $X^{*}$ is separable

**Proof :**

Let $M=\{f \in X^{*} : \|f\|=1\} \Rightarrow \quad M$ is subspace of $X^{*}$

Since $X^{*}$ is separable , then $M$ is separable, $M$ contains a countable dense subset ,
say $A=\{f_1, f_2, \ldots, f_n, \ldots\}$ , $\overline{A}=M$

since $A \subseteq M \quad \Rightarrow \quad f_n \in M$ for all $n \quad \Rightarrow \quad \|f_n\|=1$ for all $n$ .

Since $\|f_n\| = \sup\{|f_n(x)| : \|x\| = 1\}$ for all $n$ .there must exist some vector $x_n$ with $\|x_n\| = 1$ such that $|f_n(x)| > \frac{1}{2}$ **(**If such $x_n$ did not exist, this would contradict the fact that $\|f\| = 1$ **)**

Let $N$ be the closed subspace in $X$ generated by the sequence $\{x_n\}$, i.e. $N = [\{x_n\}]$. We must prove $N = X$ . Suppose that $N \neq X \;\; \Rightarrow$ there exists $x_0 \in X$ such that $x_0 \notin N$ , by theorem(6.13), there exists $f \in X^*$ such that $f(x_0) \neq 0$ , $\|f\| = 1$ and $f(x) = 0$ for all $x \in N$ .

Since $\|f\| = 1 \;\; \Rightarrow \;\; f \in M$

Since $x_n \in N \;\; \Rightarrow \;\; f(x_n) = 0$ for all $n$

$\frac{1}{2} < |f_n(x_n)| = |f_n(x_n) - f(x_n)| = |(f_n - f)(x_n)| = \|(f_n - f)\|\|x_n\| = \|f_n - f\|$ (because $\|x_n\| = 1$)

$\Rightarrow \;\; \|f_n - f\| > \frac{1}{2}$ for all $n \Rightarrow \;\; _{\frac{1}{2}}(f) \cap A = \emptyset$ where $_{\frac{1}{2}}(f) = \{g : \|g - f\| < \frac{1}{2}\} \;\; \Rightarrow \;\; f \notin \overline{A}$

This contradiction (since $\overline{A} = M$ ) and so we must have $N = X$ . It then follows that the set of all linear combinations of the $x_n$'s whose coefficients are rational. $\Rightarrow \;\; X$ is separable.

## Remark

The converse of above theorem is not true , i.e. if the normed space $X$ is separable, then $X^*$ is not necessary separable , for example, if $X = \ell^1 \;\; \Rightarrow \;\; X^* = \ell^\infty$ and $\ell^1$ is separable (see example 4.4.2), but $\ell^\infty$ is not separable (see example 4.4.3).

## Theorem(4.4.5)

Let $X$ be a normed space. If $X$ is separable space and $X^*$ is not separable, then $X$ is not reflexive .

**Proof :**

Suppose $X$ is reflexive $\Rightarrow \;\; X \simeq X^{**}$

Since $X$ is separable space $\Rightarrow \;\; X^{**}$ is separable space $\Rightarrow \;\; X^*$ is separable space This contradiction .

## Remark

If $X$ is Banach then it is not necessary reflexive .for example $\ell^1$ is Banach space, but not reflexive . because $\ell^1$ is separable and $(\ell^1)^* = \ell^\infty$ is not separable.

دراسات عليا - ماجستير

**Functional Analysis تحليل دالي**

نظري : 3   مناقشة : 1   عدد الوحدات : 3

## Exercises (4)

4.1Let $X$ ,$Y$ be linear space on a field $F$ and let $f : X \to Y$ be bijective linear function. Define $\|.\|_1 : X \to \mathbb{R}$ by $\|x_1\| = \|f(x)\|_2$ for all $x \in X$ . Show that $\|.\|_1$ is a norm on $X$ if $\|.\|_2$ is a norm on $Y$

4.2 Let $X$ be a normed space and $f$ be nonzero linear functional on $X$ . Show that either $\ker(f)$ is closed or $\ker(f)$ is dense in $X$ .

4.3 Show that : If $X$ is a locally convex space, then $X^*$ separate points on $X$ .

4.4 let $X$ and $Y$ be topological linear spaces and $f : X \to Y$ be a bijection linear function . prove or disprove $f$ is continuous iff $f^{-1}$ is also continuous

4.5 Suppose $X$ and $Y$ are topological vector spaces, $\dim Y < \infty$ , $f : X \to Y$ is linear and $f(X) = Y$ . If $\ker(f)$ is closed. Prove that $f$ is continuous.