دراسات عليا - ماجستير

Functional Analysis تحليل دالي

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9. Fixed Point Theorems and Their Applications

Fixed point theory serves as an essential for various branches of mathematical analysis and its applications. Loosely speaking, there are main approaches in this theory, metric, the topological and order – theoretic approach, where representative examples of these are Banach's, Brouwer's and Schauder -Tychonoff theorems respectively.

9.1 Definitions and General Properties

Definition(9.1.1)

Let X be a non-empty set and $f : X \to X$ be a function. A fixed point of f is simply a point $x \in X$ such that $f(x) = x$. In other words, a fixed point of f is nothing but a solution of the functional equation $f(x) = x$, $x \in X$. Fixed point have long been used in analysis to solve various kinds of integral and differential equations.

Examples(9.1.2)

(1) Let $X = \{a,b\}$.The function $f : X \to X$ defined by $f(a) = b, f(b) = a$ has no fixed point, but other three functions that function X into itself each have one or two fixed points.

(2)Let $X = [0,1]$. The graph of a function $f : X \to X$ is a subset of the unit square $X \times X$. If f is continuous, then its graph is a curve from the left edge of the square to the right edge. A point of f is an element of $[0,1]$ at which the graph of f intersects the 45^0-line.Intuitively, it seems clear that if f is continuous then it must have a fixed point (its graph must cross or touch the 45^0-line), and also that discontinuous functions f may not have a fixed point.

Definition(9.1.3)

Let (X,d) be a metric space . A function $f : X \to X$ is said to satisfy a Lipschitz condition on X if there exists a constant $k > 0$ such that $d(f(x), f(y)) \le k\,(d(x,y))$ for all $x, y \in X$. Such k is called Lipschitz constant.

- If this conditions is satisfied with Lipschitz constant k such that $0 \le k < 1$, we sat that f is a contraction function. i.e. A function $f : X \to X$ is called a contraction function if there exists a constant k**,** $0 \le k < 1$, such that $d(f(x), f(y)) \le k\, d(x,y)$ for all $x, y \in X$. Such k is called a contraction modulus of f .
- If this conditions is satisfied with Lipschitz constant k such that $k = 1$, we sat that f is a nonexpansive function. i.e. A function $f : X \to X$ is called a nonexpansive function if

$$d(f(x), f(y)) \le d(x,y) \text{ for all } x, y \in X .$$

Note that

For notational purposes we define $f^{n}(x)$, $x \in X$, $n = 0,1,2,\cdots$, inductively by $f^{0}(x) = x$ and $f^{n+1}(x) = f(f^{n}(x))$

If $f : X \to X$ is satisfy a Lipschitz condition with Lipschitz constant k, we have

$d(f^{n}(x), f^{n}(y)) \le kd(f^{n-1}(x), f^{n-1}(y)) \le \cdots \le k^{n-1}d(f(x), f(y)) \le k^{n}d(x,y)$ for all $x, y \in X$

Also $d(f^{n}(x), f^{n+1}(y)) \le kd(f^{n-1}(x), f^{n}(y)) \le \cdots \le k^{n-1}d(f(x), f^{2}(y)) \le k^{n}d(x, f(y))$ for all $x, y \in X$

Examples(9.1.4)

(1) In usual metric space $(\mathbb{R}^2, d_u)$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = \frac{1}{2}x$ for all $x \in \mathbb{R}^2$. Then f is a contraction function on $\mathbb{R}^2$

Ans :

Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$

$$d(f(x), f(y)) = d(\frac{1}{2}x, \frac{1}{2}y) = d((\frac{1}{2}x_1, \frac{1}{2}x_2), (\frac{1}{2}y_1, \frac{1}{2}y_2)) = \frac{1}{2}\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \frac{1}{2}d(x,y)$$

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable real function. If there is a real number $k < 1$ for which the derivative f' satisfies $|f'(x)| \le k$ for all $x \in \mathbb{R}$, then f is a contraction with respect to the usual metric on $\mathbb{R}$ and k is called a contraction modulus of f . This is a straightforward consequence of the mean value theorem : let $x, y \in \mathbb{R}$ with $x < y$; the mean value theorem tells us there is a number $c \in (x,y)$ such that $f(y) - f(x) = f'(c)(y - x)$ and therefore

$$|f(y) - f(x)| = |f'(c)(y - x)| = |f'(c)||y - x| \le k|y - x|$$

The sane mean value theorem argument establishes that if $k < 1$ and $f : (a,b) \to (a,b)$ satisfies $|f'(x)| \le k$ for all $x \in (a,b)$, then f is a contraction of (a,b).

Remark

Recall that a function f form metric space (X,d) into a metric space (Y,d^*) is called an uniformly continuous if for every £ > 0 there is a ¢ > 0 such that

for all $x, y \in X : d(x,y) <$ ¢ $\Rightarrow d^*(f(x),(y)) <$ £

Theorem(9.1.5)

Every contraction function is uniformly continuous.

Proof :

Let $f : X \to X$ is a contraction function on a metric space (X,d), with modulus k .

Let £ > 0, take ¢ $=$ £

Since f is contraction with modulus k, then $d(f(x), f(y)) \le k(d(x,y))$ for all $x, y \in X$ .Then $d(x,y) <$ ¢ $\Rightarrow d(f(x), f(y)) \le$ ¢$k <$ £ for all $x, y \in X$ .therefore f is uniformly continuous.

Corollary (9.1.6)

Every contraction function is continuous.

Theorem(9.1.7) Banach's contraction principle.

Let (X,d) be a complete metric space and let $f:X\to X$ is a contraction function with Lipschitzian constant k. Then f has a unique fixed point $x_0\in X$. Furthermore, for any $x\in X$ we have $\lim\limits_{n\to\infty} f^n(x)=x_0$ and $d(f^n(x),x_0)\le \dfrac{k^n}{1-k}d(x,f(x))$

Proof :

Let $x\in X$. we first show that $\{f^n(x)\}$ is a Cauchy sequence .

Since $d(f^n(x),f^{n+1}(x))\le k^n d(x,f(x))$ for all $x\in X$

Thus for $m>n$ where $n=0,1,2,\cdots$

$$d(f^n(x),f^m(x))\le d(f^n(x),f^{n+1}(x))+d(f^{n+1}(x),f^{n+2}(x))+\cdots+d(f^{m-1}(x),f^m(x))=\sum_{i=0}^{m-n-1}d(f^{n+i}(x),f^{n+i+1}(x))$$

since $\displaystyle\sum_{i=0}^{m-n-1}d(f^{n+i}(x),f^{n+i+1}(x))\le\sum_{i=0}^{m-n-1}k^{n+i}d(x,f(x))=k^n(\sum_{i=0}^{m-n-1}k^i)d(x,f(x))=\frac{k^n}{1-k}d(x,f(x))$

$\Rightarrow\ d(f^n(x),f^m(x))\le\dfrac{k^n}{1-k}d(x,f(x))$.

That is for $m>n$ where $n=0,1,2,\cdots$, we have $d(f^n(x),f^m(x))\le\dfrac{k^n}{1-k}d(x,f(x))$ as $n,m\to\infty$

$\Rightarrow\ \{f^n(x)\}$ is a Cauchy sequence in X

Since (X,d) is a complete metric space, then there is $x_0\in X$ such that $\lim\limits_{n\to\infty}f^n(x)=x_0$.

Moreover the continuity of f yields $x_0=\lim\limits_{n\to\infty}f^{n+1}(x)=\lim\limits_{n\to\infty}f(f^n(x))=f(x_0)$

Finally , letting $m\to\infty$, $\Rightarrow\ d(f^n(x),x_0)\le\dfrac{k^n}{1-k}d(x,f(x))$

Remark

The above theorem requires that $k<1$. If $k=1$ then f need not have a fixed point as the example $f(x)=x+1$ for $x\in\mathbb{R}$.

Theorem(9.1.8)

If $f:X\to X$ is a contraction function on a metric space (X,d), with modulus k, then for any $x\in X$, $d(f^n(x),x_0)\le k^n(d(x,x_0))$ for $n\in\mathbb{N}$, where x_0 is the unique fixed point of f .

Proof :

Since x_0 is a fixed point of f , then $f(x_0)=x_0$

$$d(f^n(x),x_0)=d(f(f^{n-1}(x)),f(x_0))\le kd(f^{n-1}(x),x_0)=kd(f(f^{n-2}(x),f(x_0))\le k^2d(f^{n-2}(x),x_0)$$
$$\le\cdots\le k^n d(f^0(x),x_0)=k^n d(x,x_0)$$

Theorem(9.1.9)

Let $f:X\to X$ be a continuous function on a complete metric space (X,d), and let f^m is a contraction function on X for some positive integer m .then f has a unique fixed point.

Proof :

By assumption, $g = f^m$ is a contraction on X , that is, $d(g(x), g(y)) \le kd(x,y)$ for all $x, y \in X$; here $k < 1$. Hence for every $x_0 \in X$,

$d(g^n f(x_0), g^n(x_0)) \le kd(g^{n-1} f(x_0), g^{n-1}(x_0)) \cdots \le k^n d(f(x_0), x_0) \to 0$ as $n \to \infty$

Fixed point theorem implies that g has a unique fixed point, call it x , and $g^n(x_0) \to x$. Since the mapping f is continuous, this implies $g^n f(x_0) = fg^n(x_0) \to f(x)$ .Hence $d(g^n f(x_0), g^n(x_0)) \to d(f(x), x)$ so that $d(f(x), x) = 0$. This shows that x is a fixed point of f . Since every fixed point of f is also a fixed point of g , we see that f cannot have more than one fixed point.

Theorem(9.1.10)

Let (X,d) be a complete metric space and let $f : B_r(x_0) \to X$ is a contraction function with Lipschitzian constant k (i.e. $d(f(x), f(y)) < kd(x,y)$ for all $x, y \in B_r(x_0)$) and $d(f(x_0), x_0) < (1-k)r$. Then f has a unique fixed point in $B_r(x_0)$.

Proof :

There exists r_0 with $0 \le r_0 \le r$ with $d(f(x_0), x_0) < (1-k)r_0$

We will show that $f : B_{r_0}(x_0) \to B_{r_0}(x_0)$. To see this note that if $x \in B_{r_0}(x_0)$

$d(f(x), x_0) \le d(f(x), f(x_0)) + d(f(x_0), x_0) \le kd(x, x_0) + (1-k)r_0 \le r_0$

We can apply theorem (9.1.7) to deduce that f has a unique fixed point in $B_{r_0}(x_0) \subseteq B_r(x_0)$.

Again it is easy to see that f has only one fixed point in $B_r(x_0)$.

Definition(9.1.11)

A topological space X has the fixed point property if every continuous $f : X \to X$ has a fixed point.

Theorem(9.1.12)

If X has the fixed point property and X is homeomorphic to Y ,then Y has the fixed point property.

Proof :

Let $g : Y \to Y$ be a continuous function .we want to show that g has a fixed point in Y

Since X is homeomorphic to Y , then there exists a homeomorphism function $f : X \to Y$, i.e. f is a bijective function and f, f^{-1} are continuous.

Since $f : X \to Y$, $g : Y \to Y$ and $f^{-1} : Y \to X$ are continuous functions, then $f^{-1} \circ g \circ f : X \to X$ is continuous function.

Since X has the fixed point property , then there is $x_0 \in X$ such that $(f^{-1} \circ g \circ f)(x_0) = x_0$

Hence $g(y_0) = y_0$, where $y_0 = f(x_0)$.

Definition (9.1.13)

A subset A of a topological space X is a retract of X if there is a continuous function $r : X \to A$ with $r(a) = a$, for all $a \in A$. The function r is called a retraction.

Theorem(9.1.14)

If X has the fixed point property and A is a retract of X, then A has the fixed point property.

Proof :

Let $f : A \to A$ be a continuous function .we want to show that f has a fixed point in A. Since A is a retract of X, there exists a continuous function $r : X \to A$ with $r(a) = a$, for all $a \in A$.

$\Rightarrow \quad f \circ r : X \to A \subseteq X$ is continuous function

Since X has the fixed point property, there exists $x_0 \in X$ such that $(f \circ r)(x_0) = x_0$

$\Rightarrow \quad f(r(x_0)) = x_0$. Since $f(r(x_0)) \in A \quad \Rightarrow \quad x_0 \in A$

Since $r(a) = a$, for all $a \in A$, then $r(x_0) = x_0 \;\Rightarrow\; f(x_0) = x_0$. Consequently $f(x_0) = x_0$, $x_0 \in A$

9.2 Fixed Point Theorem in Normed spaces

In section(9.1), we proved fixed point theorems in metric spaces without any algebraic structure. We now consider spaces with a linear structure but non linear functions in them. In this section we restrict our attention to normed spaces.

Theorem(9.2.1)

Let A be a nonempty, closed, convex subset of a normed space X with $f : A \to A$ nonexpansive and $f(A)$ a subset of a compact set of A. Then f has a fixed point.

Proof :

Let $x_0 \in A$. For $n = 2, 2, \cdots$, define $f_n = (1 - \frac{1}{n})f + \frac{1}{n} x_0$

Since A is convex and $x_0 \in A$, we see that $f_n : A \to A$ and it is clear that $f_n : A \to A$ is a contraction. Therefore by theorem (9.1.7) each f_n has a unique fixed point $x_n \in A$, that is,

$$x_n = f_n(x_n) = (1 - \frac{1}{n})f(x_n) + \frac{1}{n} x_0$$

In addition, since $f(A)$ lies in a compact subset of A, there exists a subsequence S of integers and $x \in A$ with $f(x_n) \to x$ as $n \to \infty$ in S.

Thus $x_n = (1 - \frac{1}{n})f(x_n) + \frac{1}{n} x_0 \to x$ as $n \to \infty$ in S.

By continuity $f(x_n) \to f(x)$ as $n \to \infty$ in S, and therefore $x = f(x)$.

Theorem(9.2.2)

The closed unit ball B^n, in $\mathbb{R}^n$, has the fixed point property, i.e. $B^n = \overline{B}_1(0) = \{x \in \mathbb{R}^n : \|x\| \le 1\}$.

Theorem (9.2.3)

Every nonempty, closed, convex sunset A of $\mathbb{R}^n$ is a retract of $\mathbb{R}^n$.

Proof :

For any $x \in \mathbb{R}^n$. Define $r : \mathbb{R}^n \to A$ such that $\|x - r(x)\| = d(x, A)$ for all $x \in A$,
Since $d(x, A) = \inf\{\|x - z\| : z \in A\}$, then there exists a unique $y = r(x) \in A$ with $\|x - y\| = \inf\{\|x - z\| : z \in A\} \Rightarrow r$ is a retraction function from $\mathbb{R}^n$ to A.

Theorem (9.2.4)

Every nonempty, bounded, closed, convex subset A of $\mathbb{R}^n$ has the fixed point property.

Proof :

Notice that A is a subset of some ball B^* in $\mathbb{R}^n$.
Since B^n and B^* are homeomorphic, theorem(9.1.11),and theorem(2.9.2) guarantee that B^* has the fixed point property. In addition, theorem(2.9.3) implies that A is a retract of B^* and therefore theorem(9.1.13) ensure that A has the fixed point property.

Corollary(9.2.5)

Every nonempty, bounded, closed, convex subset A of a finite dimensional normed space has the fixed point property.

Proof :

Since any finite dimensional normed space X is isomorphic to $\mathbb{R}^n$ with $n = \dim(X)$, we have Every nonempty, bounded, closed, convex subset A of a finite dimensional normed space has the fixed point property.

We would like to extend theorem(9.2.4) to a infinite dimensional space.

Example(9.2.6)

Let $X = \ell^2$, $B = \{x \in \ell^2 : \|x\| \le 1\}$. $\partial(B) \subseteq B$ .define $f : B \to \partial(B)$ by $f(x) = \sqrt{1 - \|x\|^2}$ for all $x \in B$
It is easy to see that f is continuous but not have a fixed point.

Theorem (9.2.7) Brouwer fixed point theorem

Let K be a nonempty, compact, convex subset of a finite dimensional normed space. Then K has the fixed point property(i.e. every continuous function $f : K \to K$ has fixed point).

Proof :

Remark

The Brouwer theorem requires only that f be continuous, not that it be a contraction, so there are lots of situations in which the Brouwer theorem applies but the fixed theorem doesn't. in particular, Brouwer's theorem confirms our intuition that any continuous function from $[0,1]$ into itself has a fixed point, not just the functions the functions that satisfy $|f'(x)| \le k$ for some $k < 1$.
But conversely, the fixed point theorem doesn't require compactness or convexity , in fact, it doesn't require that the domain of f be a subset of linear space, as this version of Brouwer's theorem does. So there are also lots of situations where fixed point theorem applies and Brouwer's doesn't.

Definition(9.2.8)

Let X and Y be normed spaces .

(1) A function $f : X \to Y$ is called compact if $f(X)$ contained in a compact subset of Y .

(2) A function $f : X \to Y$ is called completely continuous if it is both continuous and compact.

(3) A compact function $f : X \to Y$ is called finite dimensional if $f(X)$ contained in a finite dimensional subspace of Y .

Note that

Let A be a subset of a normed space X . A function $f : A \to X$ is compact if $f(B)$ is a compact subset of X whenever B is bounded subset of A .

Remark

We next extend Brouwer's fixed point theorem to compact map in normed spaces. This generalization is due to Schauder. The main idea is to approximate compact functions by functions with finite dimensional ranges.

Let $D = \{x_1, x_2, \cdots, x_n\}$ be a finite subset of a normed space X and for fixed $£ > 0$ let

$D_{£} = \bigcup_{i=1}^{n} {}_{£}(x_i)$ where ${}_{£}(x_i) = \{x \in X : \|x - x_i\| < £\}$

For each $i = 1, 2, \cdots, n$, let $«_i : D_{£} \to \mathbb{R}$ given by $«_i(x) = \max\{0, £ - \|x - x_i\|\}$ for all $x \in D_{£}$.

The Schauder Projection is the function $P_{£} : D_{£} \to conv(D)$ given by

$$P_{£}(x) = \frac{\sum_{i=1}^{n} «_i(x) x_i}{\sum_{i=1}^{n} «_i(x)}$$ for all $x \in D_{£}$.

Notice $P_{£}$ is well defined . since if $x \in D_{£}$, then $x \in {}_{£}(x_i)$ for some $i \in \{1, 2, \cdots, n\}$ and therefore $\sum_{i=1}^{n} «_i(x) \neq 0$.

Also $P_{£}(x) \subseteq conv(D)$ since each $P_{£}(x)$ is convex combination of the points $x_1, x_2, \cdots, x_n$

Theorem (9.2.9)

Let A be a convex subset of a normed space X , and $D = \{x_1, x_2, \cdots, x_n\} \subseteq A$ .If $P_{£}$ denoted the Schauder Projection, then

(1) $P_{£}$ is a convex, continuous function from $D_{£}$ into $conv(D) \subseteq A$, and

(2) $\|x - P_{£}\| < £$ for all $x \in D_{£}$.

Proof :

Theorem (9.2.10)

Let A be a convex subset of a normed space X , and $f : X \to A$ a compact, continuous function. Then for each $£ > 0$,there are a finite set $D = \{x_1, x_2, \cdots, x_n\}$ in $f(X)$ and a finite dimensional continuous function $f_£ : X \to A$ with the following properties

(1) $\|f_£(x) - f(x)\| < £$ for all $x \in X$ (2) $f_£(x) \subseteq conv(D) \subseteq A$

Proof :

Theorem (9.2.11)

Let A be a closed subset of a normed space X , and $f : X \to A$ a compact, continuous function. Then f has fixed point iff $£ - f$ has fixed point

Proof :

Theorem (9.2.12)

Let f be a completely continuous of a normed space X into itself and let $f(X)$ be bounded . Then f has fixed point.

Proof :

We now state and prove Schauder's fixed point theorem

Theorem (9.2.13) Schauder fixed point theorem

Let K be a nonempty, closed, convex subset of normed linear space X .Let f be a continuous function of K into a compact subset of K .Then f has fixed point in K .

Proof :

Theorem (9.2.14)

Let K be a nonempty, compact, convex subset of normed linear space X . Every continuous function $f : K \to K$ has fixed point.

Proof :

Remark

This theorem would apply, for example, to any compact convex subset of $C[0,1]$,the linear space of continuous functions on the unit interval, with the max norm.

Definition(9.2.15)
A norm $\|.\|$ on a linear space X is said to be strictly convex if $\|x+y\|=\|x\|+\|y\|$ only when x and y linearly independent.

Theorem(9.2.16) Clarkson
If a normed space X has a countable everywhere dense subset, then there exists a strictly convex norm on X equivalent to the given norm.

Proof :
Let $A=\{x \in X : \|x\|=1\}$

Theorem(9.2.17) Clarkson
Let K be a compact convex subset of a normed space X with a strictly convex norm. Then to each point x of X corresponds a unique point Px of K at K at minimum distance from x, i.e.
$\|x-Px\|=\inf\{\|x-y\|: y \in K\}$ and the function $x \to Px$ is continuous in X. The function P is called the metric projection onto K.

Proof :
Let $x \in X$

9.3 Fixed Point Theorem in Hausdorff Locally convex spaces

This section presents fixed point results for functions defined on Hausdorff Locally convex spaces. We begin with the Schauder-Tychonoff theorem with is an extension of Schauder's fixed point theorem, which in turn is an extension of Brouwer's fixed point theorem. In the proof we will need the following approximation theorem.

Theorem () Schauder – Tychonoff Theorem
Let K be a non-empty compact convex subset of a locally convex Hausdorff space X, and let $f: X \to X$ be continuous function. Then f has a fixed point in X.

Theorem (9.3.1) Schauder – Tychonoff Theorem
Let K be a closed convex subset of a locally convex Hausdorff space X, and let f be a continuous function of K into a compact subset of K. Then f has a fixed point in X.

9.4 Fixed Point Theorem in Ordered Linear spaces

A relation R on a set X is said to be a partial order relation if satisfies the following three conditions
(1) xRx, for every $x \in X$ (reflexivity) (2) xRy and yRz implies xRz (transitivity)
(3) xRy and yRx implies $x=y$ (antisymmetry)

We shall denote the partial order relation by the symbol $\leq$. The set X together with the partial order i.e. the pair $(X,\leq)$ is called a partially ordered set. If $x \leq y$, $(x, y \in X)$, we say that x precedes or smaller than y and that y follows or larger than x. The symbol $x < y$ stands for $x \le y$ and $x \ne y$ and reads " strictly precedes y " or " strictly dominates x ".

Two elements x and y of partially ordered set are said to be **comparable** if one of them is smaller than or equal to the other, i.e. if $x \le y$ or $y \le x$. A partial ordering $\le$ on a set X is called total (or linear or simple or complete) ordering iff either $x \le y$ or $y \le x$ for every pair $x, y \in X$. A set with a total ordering is called a totally ordered set or a chain. An element $a \in X$ is called the first or the smallest element of X, if $a \le x$ for every $x \in X$. similarly an element $b \in X$ is called the last or the largest element of X, if $x \le b$ for every $x \in X$. An element $a \in X$ is called a minimal element of X, if $x \le a$ implies $a = x$. similarly an element $b \in X$ is called a maximal element of X, if $b \le x$ implies $b = x$.

An element $a \in X$ is said to be a lower bound of A if $a \le x$ for every $x \in A$. Similarly an element $b \in X$ will be an upper of A if $x \le b$ for every $x \in A$. A set A may have no lower bounds or no upper bounds, or it may have many. Let A^* denote the collection of all upper bounds of A and A_* denote the collection of all lower bounds of A. The smallest member of A^*,if it exists, is called the least upper bound (l.u.b)or the supremum of A written as $\sup A$. Similarly the largest member of A_*, if it exists, is called the greatest lower bound (g.l.b) of infimum of A written as $\inf A$. Note that the $\inf A$ and the $\sup A$ may or may not be members of A.

A partially ordered set X is said to be will ordered if every subset of X contains a first element. Let A be a subset of a partially ordered set X.

Definition (9.4.1)

Let X be a real linear space. A partial order relation $\le$ on X is call linear order if the following axioms are satisfied

(1) $x \le y \;\Rightarrow\; x + z \le y + z$ for all $x, y, z \in X$

(2) $x \le y \;\Rightarrow\; \alpha x \le \alpha y$ for all $x, y \in X$ for all $\alpha \ge 0$

A real linear space endowed with a linear order is called an ordered linear space. An element x of an ordered linear space X is said to be positive if $x \ge 0$, and negative if $x \le 0$. The set of all positive elements of an ordered linear space X with be denoted by X_+, i.e. $X_+ = \{x \in X : x \ge 0\}$

Definition(9.4.2)

A subset A of real linear space X is called a positive cone if it satisfies

(1) $x, y \in A \;\Rightarrow\; x + y \in A$ (2) $x \in A$ and $\alpha \ge 0 \;\Rightarrow\; \alpha x \in A$

(3) $x, -x \in A \;\Rightarrow\; x = 0$ (4) A contains non-zero element.

It is clear to show that X_+ is a positive cone of X.

It is easy to show that X_+ is a convex cone of X, i.e. $X_+ + X_+ \subseteq X_+$ and $\alpha X_+ \subseteq X_+$.

9.5 Some Applications of Fixed Point Theorems

Fixed point theorem has important application
Such theorems are most important tools for proving the existence and uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations, and variational inequalities, etc.)

Application of Fixed Point Theorem to Linear Equations

Fixed point theorem has important application to iteration methods for solving systems of linear algebraic equations and yields sufficient conditions for convergence and error bounds.
Suppose we want to find the solution of system on n linear algebraic equation with n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

This system can be written as $Ax = b$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{bmatrix} = [a_{ij}], \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_i), \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = (b_i)$$

To apply fixed point theorem, we need a complete metric space and a contraction function on it. Let $X = \mathbb{R}^n$ with metric d given by $d(x,y) = \max_i |x_i - y_i| \quad \cdots(1)$

where $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in X$, then (X, d) is complete metric space. Define $f : X \to X$ by $y = f(x) = Ax + b \quad \cdots(2)$

where $A = [a_{ij}]$ is a fixed real $n \times n$ matrix and $b \in X$ a fixed vector. under what condition will f be a contraction , we have

$$y_i = \sum_{j=1}^{n} a_{ij} x_l + b_i$$

Setting $w = (w_i) = f(z)$, we thus obtain from(1) and (2)

$$d(y,w) = d(f(x), f(z)) = \max_i |y_i - w_i| = \max_i \left| \sum_{j=1}^{n} a_{ij}(x_i - z_i) \right| \le \max_i |x_i - z_i| \left| \sum_{j=1}^{n} a_{ij} \right| = d(x,z) \max_i \left| \sum_{j=1}^{n} a_{ij} \right|$$

We see that this can be written $d(y,w) \le kd(x,z)$, where $k = \max\limits_{i} \left| \sum_{j=1}^{n} a_{ij} \right|$. Fixed point theorem thus yields.

Theorem (9.5.1) Linear equation

If a system $x = Ax + b$, $(A = [a_{ij}], b$, be given), of n linear equations in n unknowns $x_1, x_2, \cdots, x_n$ (the components of x) satisfies $\sum_{j=1}^{n} |a_{ij}| < 1$, $i = 1, 2, \cdots, n$ it has precisely one solution x .

Application of Fixed Point Theorem to Differential Equations

The most interesting applications of fixed point theorem arise in connection with function spaces. The theorem then yields existence and uniqueness theorems for differential and integral equations, as we shall see. In fact, in this section let us consider an explicit ordinary differential equation of the first order

$$y' = \frac{dy}{dx} = f(x,y)$$

An initial value problem for such an equation consists of the equation and initial condition $y(x_0) = y_0$, where x_0 and y_0 are given real number.

We shall use fixed point theorem to prove the famous Picard's theorem

Theorem (9.5.2) Picard Theorem

Let D denote an open set in $\mathbb{R}^2$, $(x_0, y_0) \in D$. Let f be real valued function defined and continuous in D, and let it satisfy Lipschitz condition of the form

$$|f(x,y_1) - f(x,y_2)| \le k\,|y_1 - y_2|, \qquad (x,y_1),(x,y_2) \in D$$

in the variable y . Then there is an interval $|x - x_0| < \delta$ in which the differential equation $\frac{dy}{dx} = f(x,y)$ has a unique solution $y = \varphi(x)$ satisfying the initial condition $\varphi(x_0) = y_0$

proof :

Together the differential equation $y' = f(x,y)$ and the initial condition $\varphi(x_0) = y_0$ are equivalent to the integral equation

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt$$

By the continuity of f , we have $|f(x,y)| < M$ in some domain $D' \subset D$ containing the point (x_0, y_0). Choose $\delta > 0$ such that (1) $(x,y) \in D'$ if $|x - x_0| < \delta$, $|y - y_0| < M\delta$ (2) $k\delta < 1$

and let C^* be the space of continuous functions ¨ defined on the interval $|x - x_0| < ¢$ and such that $|¨(x) - y_0| < M¢$,equipped with the metric d defined by $d(¨, ¨') = \max_x |¨(x) - ¨'(x)|$.
The space C^* is complete, since it is closed subspace of the space of all continuous functions on $[x_0 - ¢, x_0 + ¢]$.defined $g : C^* \to C^*$ by $g(¨) = ¤$, where

$$¤(x) = y_0 + \int_{x_0}^{x} f(t, ¨(t))dt, \qquad |x - x_0| < ¢$$

If $¨ \in C^*$, $|x - x_0| < ¢$, then $|¤(x) - y_0| = \left|\int_{x_0}^{x} f(t, ¨(t))dt\right| \le \int_{x_0}^{x} |f(t, ¨(t))|dt$

Since $|f(x, y)| < M$, then $\int_{x_0}^{x} |f(t, ¨(t))|dt < M|x - x_0| < M¢$, so that $|¤(x) - y_0| < M¢$, hence $¤ \in C^*$. Moreover,

$$|¤(x) - ¤'(x)| = \left|\int_{x_0}^{x} (f(t, ¨(t)) - f(t, ¨'(t)))dt\right| \le \int_{x_0}^{x} |f(t, ¨(t)) - f(t, ¨'(t))|dt \le k¢ \max_x |¨(x) - ¨'(x)| \le k¢ d(¨, ¨')$$

After maximizing with respect to x .But $k¢ < 1$, so that g is contraction function. It follows from fixed point theorem that equation $g(¨) = ¨$, i.e. the integral equation

$$¨(x) = y_0 + \int_{x_0}^{x} f(t, ¨(t))dt$$

has a unique solution in the space C^* .

Remark

Picard theorem can easily be generalized to the case of systems of differential equations :

Theorem(9.5.3)generalized Picard theorem

Let D denote an open set in $\mathbb{R}^{n+1}, (x_0, y_{01}, y_{02}, \cdots, y_{0n}) \in D$. Let f_i be functions defined and continuous in D , and let it satisfy Lipschitz condition of the form

$$|f(x, y_1, y_2, \cdots, y_n) - f(x, y_1', y_2', \cdots, y_n')| \le k \max_{1 \le i \le n} |y_i - y_i'|, \qquad (x, y_1), (x, y_2) \in D$$

in the variables $y_1, y_2, \cdots, y_n$. Then there is an interval $|x - x_0| < ¢$ in which the system of differential equations $\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n), \quad i = 1, 2, \cdots, n$ has a unique solution $y_1 = ¨_1(x),\ y_2 = ¨_2(x), \cdots, y_n = ¨_n(x)$ satisfying the initial condition
$¨_1(x_0) = y_{01}, ¨_2(x_0) = y_{02}, \cdots, ¨_n(x_0) = y_{0n}$

proof :

Together the differential equations $\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n), \quad i = 1, 2, \cdots, n$ and the initial conditions $¨_1(x_0) = y_{01}, ¨_2(x_0) = y_{02}, \cdots, ¨_n(x_0) = y_{0n}$ are equivalent to the system of integral equation

$$¨_i(x) = y_{0i} + \int_{x_0}^{x} f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t))dt, \quad i = 1, 2, \cdots, n$$

By the continuity of the functions f_i, we have $|f(x, y_1, y_2, \cdots, y_n)| < M, \quad i = 1, 2, \cdots, n$ in some domain $D' \subset D$ containing the point $(x_0, y_{01}, y_{02}, \cdots, y_{0n})$. Choose $¢ > 0$ such that
(1) $(x, y_1, y_2, \cdots, y_n) \in D'$ if $|x - x_0| < ¢, \quad |y_i - y_{i0}| < M¢$ for all $i = 1, 2, \cdots, n$ (2) $k¢ < 1$
This let C^* be the space of ordered n - tuples $¨ = (¨_1, ¨_2, \cdots, ¨_n)$ of continuous functions $¨_1, ¨_2, \cdots, ¨_n$ defined on the interval $|x - x_0| < ¢$ and such that $|¨_i(x) - y_{0i}| < M¢$ for all $i = 1, 2, \cdots, n$, equipped with the metric d defined by $d(¨, ¨') = \max_x |¨_i(x) - ¨'_i(x)|$. The space C^* is complete, since it is closed subspace of the space of all continuous functions on $[x_0 - ¢, x_0 + ¢]$.defined $g : C^* \to C^*$ by $g(¨) = ¨$, where $¤ = (¤_1, ¤_2, \cdots, ¤_n)$

$$¨_i(x) = y_{0i} + \int_{x_0}^{x} f_i(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t))dt, \qquad |x - x_0| < ¢, \quad i = 1, 2, \cdots, n$$

If $¨ = (¨_1, ¨_2, \cdots, ¨_n) \in C^*$, $|x - x_0| < ¢$, then

$$|¤_i(x) - y_{0i}| = \left| \int_{x_0}^{x} f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t))dt \right| \le \int_{x_0}^{x} |f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t))| dt, \quad i = 1, 2, \cdots, n$$

Since $|f(x, y_1, y_2, \cdots, y_n)| < M$, then $\int_{x_0}^{x} |f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t))| dt < M\,|x - x_0| < M¢$, so that $|¤_i(x) - y_{0i}| < M¢, \quad i = 1, 2, \cdots, n$, hence $¤ = (¤_1, ¤_2, \cdots, ¤_n) \in C^*$. Moreover,

$$|¤_i(x) - ¤'_i(x)| = \left| \int_{x_0}^{x} (f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t)) - f(t, ¨'_1(t), ¨'_2(t), \cdots, ¨'_n(t))dt \right| \le \int_{x_0}^{x} |f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t) - f(t, ¨'_1(t), ¨'_2(t), \cdots, ¨'_n(t)| dt$$

$$\le k¢ \max_x |¨_i(x) - ¨'_i(x)| \le k¢ d(¨, ¨')$$

After maximizing with respect to x .But $k¢ < 1$, so that g is contraction function. It follows from fixed point theorem that equation $g(¨) = ¨$, i.e. the integral equation

$$¨_i(x) = y_{0i} + \int_{x_0}^{x} f(t, ¨_1(t), ¨_2(t), \cdots, ¨_n(t))dt, \quad i = 1, 2, \cdots, n$$

has a unique solution in the space C^* .

Application of Fixed Point Theorem to Integral Equations

We now show how the method of successive approximations can be used to prove the existence and uniqueness of solutions of integral equations. An integral equation of the form

$$f(x) = \varphi(x) + \lambda \int_a^b k(x,y) f(y)\,dy \qquad \cdots(1)$$

is called a **Fredholm equation** of the second kind. Here $[a,b]$ is a given interval. f is a function on $[a,b]$ which is unknown. λ is parameter. The function k is called the kernel of the equation defined on the square $D=[a,b]\times[a,b]$, and φ is a given function on $[a,b]$, the equation is said to be homogeneous if $\varphi=0$ (but otherwise non-homogeneous). An integral equation of the form

$$f(x) = \varphi(x) + \lambda \int_a^x k(x,y) f(y)\,dy \qquad \cdots(2)$$

is called a **Volterra equation**. The difference between(1) and (2) is that in (1) the upper limit of integration b is constant, where as here in (2) it is variable.

Theorem (9.5.4) Fredholm integral equation theorem

Suppose k and φ in the integral equation (1) to be continuous on $[a,b]\times[a,b]$ and $[a,b]$, respectively, and assume that λ satisfies $|\lambda| < \dfrac{1}{c(b-a)}$ with c defined in $|k(x,y)| < c, \quad x,y\in[a,b]$. Then the Fredholm integral equation has a unique solution f on $[a,b]$.

Proof :

Since k is continuous function, so that $|k(x,y)| < c, \quad a\le x\le b, \quad a\le y\le b$

Let $X=C[a,b]$, the space of all continuous functions defined on the interval $[a,b]$ with metric d given by $d(f,g)=\max\limits_{a\le x\le b}|f(x)-g(x)|$, then (X,d) is complete metric. Define

$T: X\to X$ by $T(f)=g$ where $g(x)=\varphi(x)+\lambda\int_a^b k(x,y)f(y)\,dy$

If $T(f_1)=g_1$ and $T(f_2)=g_2$, then

$$g_1(x)-g_2(x)=\varphi(x)+\lambda\int_a^b k(x,y)f_1(t)\,dt-\left(\varphi(x)+\lambda\int_a^b k(x,y)f_2(t)\,dt\right)=\lambda\int_a^b k(x,y)(f_1(t)-f_2(t))\,dt$$

$$d(g_1,g_2)=\max_{a\le x\le b}|g_1(x)-g_2(x)|=\max_{a\le x\le b}\left|ª\int_a^b k(x,y)(f_1(y)-f_2(y))dy\right|\le|ª|\max_{a\le x\le b}\int_a^b|k(x,y)(f_1(y)-f_2(y))|dy$$

$$\le|ª|\max_{a\le x\le b}\int_a^b|k(x,y)||f_1(y)-f_2(y)|dy\le|ª|c\max_{a\le x\le b}|f_1(x)-f_2(x)|\left|\int_a^b dy\right|=|ª|c(b-a)\max_{a\le x\le b}|f_1(x)-f_2(x)|$$

$$=|ª|c(b-a)d(f_1,f_2)$$

so that ¤ is contraction function if $|ª|<\frac{1}{c(b-a)}$. It follows from fixed point theorem that the integral equation (1) has a unique solution for any value of ª satisfying $|ª|<\frac{1}{c(b-a)}$.

Remark

The successive approximations $f_0,f_1,\cdots,f_n,\cdots$ to this solution are given by

$$f_n(x)=¨(x)+ª\int_a^b k(x,y)f_{n-1}(y)dy,\quad n=1,2,\cdots$$

where any function continuous on $[a,b]$ can be chosen as f_0.Note that the method of successive approximations can be applied to the equation (1) only for sufficiently small $|ª|$

Theorem(9.5.6) Volterra integral equation

Suppose that k and ¨ in the integral equation (2) to be continuous on R and $[a,b]$, respectively, where $R=\{(x,y)\in\mathbb{R}^2 : a\le y\le x,\quad a\le x\le b\}$. Then the Volterra integral equation has a unique solution f on $[a,b]$ for every ª .

Proof :

Let $X=C[a,b]$, the space of all continuous functions defined on the interval $[a,b]$ with metric d given by $d(f,g)=\max_{a\le x\le b}|f(x)-g(x)|$, then (X,d) is complete metric.

Define $¤ : X\to X$ by $¤f(x)=¨(x)+ª\int_a^x k(x,y)f(y)dy$

Let $f,g\in X$

$$¤f(x)-¤g(x)=¨(x)+ª\int_a^x k(x,y)f(y)dy-¨(x)-ª\int_a^x k(x,y)g(y)dy=ª\int_a^x k(x,y)(f(y)-g(y))dy$$

$$|¤f(x)-¤g(x)|=\left|ª\int_a^x k(x,y)(f(y)-g(y))dy\right|\le|ª|\int_a^x|k(x,y)||f(y)-g(y)|dy$$

$$\le|ª|c\max_x|f(x)-g(x)|\left|\int_a^x dy\right|=|ª|c(x-a)\max_x|f(x)-g(x)|=|ª|c(x-a)d(f,g)$$

where $c=\max_{x,y}|K(x,y)|$

$$\left|\text{¤}^2 f(x)-\text{¤}^2 g(x)\right|=\left|\text{ª}\int_a^x k(x,y)(\text{¤}f(y)-\text{¤}g(y))dy\right|\le|\text{ª}|\int_a^x|k(x,y)||\text{¤}f(y)-\text{¤}g(y)|dy$$

$$\le|\text{ª}|c\int_a^x|\text{ª}|c(x-a)dy\,d(f,g)=|\text{ª}|^2c^2\frac{(x-a)^2}{2}d(f,g)$$

We show by induction that

Assuming $\left|\text{¤}^m f(x)-\text{¤}^m g(x)\right|\le|\text{ª}|^m c^m\dfrac{(x-a)^m}{m!}d(f,g)$ hold for any m, we obtain

$$\text{¤}^{m+1}f(x)-\text{¤}^{m+1}g(x)=\text{¨}(x)+\text{ª}\int_a^x k(x,y)\text{¤}^m f(y)dy-\text{¨}(x)-\text{ª}\int_a^x k(x,y)\text{¤}^m g(y)dy=\text{ª}\int_a^x k(x,y)(\text{¤}^m f(y)-\text{¤}^m g(y))dy$$

$$\left|\text{¤}^{m+1}f(x)-\text{¤}^{m+1}g(x)\right|=|\text{ª}|\left|\int_a^x k(x,y)(\text{¤}^m f(y)-\text{¤}^m g(y))dy\right|\le|\text{ª}|\int_a^x|k(x,y)||\text{¤}^m f(y)-\text{¤}^m g(y)|dy$$

$$\le|\text{ª}|c\int_a^x|\text{ª}|^m c^m\frac{(x-a)^m}{m!}dy\,d(f,g)=|\text{ª}|^{m+1}c^{m+1}\frac{(x-a)^{m+1}}{(m+1)!}d(f,g)$$

Which completes the inductive proof. i.e. $d(\text{¤}^n f,\text{¤}^n g)\le|\text{ª}|^n c^n\dfrac{(b-a)^n}{n!}d(f,g)$

We finally note that a Volterra equation can be regarded as a special Fredholm equation whose kernel k is zero in the part of the square $[a,b]\times[a,b]$ where $y>x$ and may not be continuous at points on the diagonal $(y=x)$.

دراسات عليا - ماجستير

Functional Analysis تحليل دالي

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Exercises (9)

9.1 Let $X = \{x \in \mathbb{R} : x \geq 1\} \subset \mathbb{R}$ and let the function $f : X \to X$ be defined by $f(x) = \frac{1}{2}x + x^{-1}$ for all $x \in X$ .Show that f is a contraction and fined the smallest k .

9.2 Let (X,d) be a compact metric space with $f : X \to X$ satisfying $d(f(x), f(y)) < d(x,y)$ for $x, y \in X$ and $x \neq y$. Show that f has a unique fixed point in X .

9.3 Let $¯_r(x_0)$ in Banach space X with $f : ¯_r(x_0) \to X$ is a contraction function and $f(\partial(¯_r(x_0)) \subseteq ¯_r(x_0)$. Show that f has a unique fixed point in $_r(x_0)$.

9.4 Let (X,d) be a complete metric space with $f : X \to X$ satisfying $d(f(x), f(y)) < ¨(d(x,y))$ where $¨ : [0,\infty) \to [0,\infty)$ is non decreasing function with $\lim_{n\to\infty} ¨^n(t) = 0$ for $t > 0$. Show that f has a unique fixed point $x_0 \in X$ with $\lim_{n\to\infty} f^n(x) = x_0$ for $x \in X$.

9.5 Let (X,d) be a complete metric space and let $f : X \to X$ be such that $f^n : X \to X$ is a contraction for positive integer n . Show that f has a unique fixed point $x_0 \in X$ and that for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x_0$.

9.6 Let G be an open subset of a Banach space X and let $f : G \to X$ be a contraction. Show that $(\mathrm{I} - f)(G)$ is open.

9.7 Let (X,d) be a complete metric space and let $¨ : X \to [0,\infty)$ be a function. Suppose $\inf\{¨(x) + ¨(y) : d(x,y) \geq ¥\} = «(¥) > 0$ for all $¥ > 0$. Show that each sequence $\{x_n\}$ in X , for which $\lim_{n\to\infty} ¨(x_n) = 0$, converges to one and only one point $x_0 \in X$.

9.8 Let (X,d) be a complete metric space and let $f : X \to X$ be a continuous. Suppose $¨(x) = d(x, f(x))$ satisfies $\inf\{¨(x) + ¨(y) : d(x,y) \geq ¥\} = «(¥) > 0$ for all $¥ > 0$, and that $\inf\{d(x, f(x)) : x \in X\} = 0$. Show that f has a unique fixed point.

9.9 Let A be a nonempty, closed, bounded, convex set in a Hilbert space X . Show that each nonexpansive function $f : A \to A$ has at least one fixed point.

9.10 Let X be a uniformly convex Banach space and A be a closed, bounded, convex subset of X . Show that each every nonexpansive function $f : A \to A$ has fixed point.

9.11 solve by iteration, choosing $x_0 = ¤$: $f(x) = ¨(x) + ª\int_0^1 e^{x-y} f(y)\,dy$

9.12 If ¤ and k are continuous on $[a,b]$ and $C = [a,b] \times [a,b] \times \mathbb{R}$,respectively, and k satisfies on $D = [a,b] \times [a,b]$ a Lipschitz condition of the form $|k(x,y,z_1) - k(x,y,z_2)| \leq M\,|z_1 - z_2|$.

Show that the nonlinear integral equation $f(x) = ¨(x) + ª\int_a^b k(x,y,f(y))\,dy$ has a unique solution f for any ª such that $|ª| < \frac{1}{M(b-a)}$.