On properties of Sugeno Fuzzy Measure space

Firas Hussean Maghool

AL- Qadisiya University

College of Computer science and Mathematic

Department of mathematics.

h.mfiras@yahoo.com

Abstract:

In this paper, first study some properties of a Sugeno fuzzy measure as a set function and Continunity of sugeno fuzzy measure space, discuss pesuduometric generating properties of this measure and prove some theory.

Keyword:- σ -field , Sugeno fuzzy measure, pesuduometric generating properties

Introduction:

Fuzzy measure is generalization of the notion of measure in mathematical analysis . In 1974, the Japanese Scholar Sugeno [1] presented a kind of typical non additive measure Sugeno fuzzy measure. The properties and application of Sugeno measure have been studied by many authors as [2]and [4] , in this paper we present definition sugeno fuzzy measure as a monotone and continuous set function and study pesuduometric generating properties of sugeno fuzzy measure space and prove some important theory

Preliminaries

Definition(2.1)[1]:

A family F of subsets of a set X is called a σ-field on a set Ω, if

(1) $X \in F$

(2) If $A \in F$, then $A^c \in F$

(3) If $A_n \in F, n = 1,2,...$ then $\bigcup_{n=1}^{\infty} A_n \in F$

A measurable space is a pair (X ,F),

where X is a set and F is σ-field on X

A subset A of X is called measurable

(measurable with respect to the

σ-field F) , if $A \in F$ i.e. any member of F is called a measurable set.

Definition(2.2)[1][2]:-

Let X is nonempty set and F is $\sigma-$ field , a set function $\mu : F \to [0,1]$ that is satisfy the following axioms:

(1) $\mu(X) = 1$

$$(2)\mu(\bigcup_{i=1}^{\infty} A_i) = \begin{cases} \frac{1}{\lambda}[\prod_{i=1}^{\infty}(1+\lambda \cdot \mu(A_i))]-1 & , \quad \lambda \neq 0 \\ \sum_{i=1}^{\infty} \mu(A_i) & , \quad \lambda = 0 \end{cases}$$

where $\lambda \in (-1,\infty)$ is called λ- Sugeno fuzzy measure and atripe (X,F,μ) λ - Sugeno fuzzy measure space , X is a set.

Example: Let $\mu : X \to [0,1]$

$X = \{1,2\}, \quad F = \{\phi, \{1\}, \{2\}, X\}$

$$\mu(A) = \begin{cases} 0, & A = \phi \\ 0.4, & A = \{1\} \\ 0.2, & A = \{2\} \\ 1 & A = X \end{cases}$$

Is λ - Sugeno fuzzy measre where $\lambda = 5$

Solve:-

$\mu(\{1\} \cup \{2\}) = \mu(X) = 1$

$\mu(\{1\} \cup \{2\}) = \frac{1}{5}\{1+5\mu\{1\})(1+5\mu(\{2\})-1\}$

$= \frac{1}{5}[(1+2)(1+1)]-1$

$= \frac{6}{5} - \frac{1}{5} = 1$

In the following we called λ- Sugeno fuzzy measure is Sugeno fuzzy measure

Main Result

1- Properties of Sugeno fuzzy measure space:

Theorem(2.3):

Let (X,F,μ) be Sugeno fuzzy measure space then

(1) $\mu(\phi) = 0$

(2) if $A_1, A_2 \in F$ then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) + \lambda\mu(A_1)\mu(A_2)$

(3) if $A_1, A_2, \ldots, A_n \in F$ then

$$\mu(\bigcup_{k=1}^{n} A_i) = \begin{cases} \frac{1}{\lambda}[\prod_{k=1}^{n}(1+\lambda\cdot\mu(A_k))-1 & , \quad \lambda \neq 0 \\ \sum_{k=1}^{n}\mu(A_k) & , \quad \lambda = 0 \end{cases}$$

(4) if $A \in F$

Then $\mu(A)+\mu(A^c) = 1-\lambda\mu(A)\mu(A^c)$

Proof :

(1) since $X \cup \phi = X$

$\mu(X \cup \phi) = \mu(X) = 1$

$\mu(X \cup \phi) = \mu(X) + \mu(\phi) + \lambda\mu(X)\mu(\phi)$

$\Rightarrow 1+\mu(\phi)+\lambda\mu(\phi) = 1$

$\Rightarrow \mu(\phi)+\lambda\mu(\phi) = 0$

$\Rightarrow (1+\lambda)\mu(\phi) = 0$

(2) $\mu(A_1 \cup A_2) = \frac{1}{\lambda}(\prod_{i=1}^{2}(1+\lambda\mu(A_i))-1)$

$= \frac{1}{\lambda}[(1+\lambda\mu(A_1)(1+\lambda\mu(A_2)-1)]$

$= \frac{1}{\lambda}[(1+\lambda\mu(A_1+\lambda\mu(A_2)+\lambda^2\mu(A_1)\mu(A_2)-1]$

$= \frac{1}{\lambda}[(\lambda\mu(A_1+\lambda\mu(A_2)+\lambda^2\mu(A_1)\mu(A_2)]$

$= \mu(A_1)+\mu(A_2)+\lambda\ \mu(A_1)\mu(A_2)$

(3) put $A = \phi$ when $k \geq n$

$$\mu(\bigcup_{k=1}^{n} A_k) = \mu(\bigcup_{k=1}^{\infty} A_k) = \begin{cases} \frac{1}{\lambda}[\prod_{k=1}^{\infty}(1+\lambda\cdot\mu(A_k))-1 & , \quad \lambda \neq 0 \\ \sum_{k=1}^{\infty}\mu(A_k) & , \quad \lambda = 0 \end{cases}$$

$$\mu(\bigcup_{k=1}^{n} A_k) = \begin{cases} \frac{1}{\lambda}[\prod_{k=1}^{n}(1+\lambda\cdot\mu(A_k)) \prod_{k=n+1}^{\infty}(1+\lambda\mu(A_k))-1] & , \quad \lambda \neq 0 \\ \sum_{k=1}^{n}\mu(A_k)+\sum_{k=1}^{\infty}\mu(A_k) & , \quad \lambda = 0 \end{cases}$$

$$\mu(\bigcup_{k=1}^{n} A_i) = \begin{cases} \frac{1}{\lambda}[\prod_{k=1}^{n}(1+\lambda\cdot\mu(A_k))-1 & , \quad \lambda \neq 0 \\ \sum_{k=1}^{n}\mu(A_k) & , \quad \lambda = 0 \end{cases}$$

(4) since $\lambda \in F, A = \phi$

Then $\mu(\phi) \subseteq \mu(A) \subseteq \mu(X) \Rightarrow 0 \leq \mu(A) \leq 1$

Since $A \cup A^c = X \rightarrow \mu(A \cup A^c) = \mu(X) = 1$

$\mu(A)+\mu(A^c)+\lambda\mu(A)\mu(A^c) = 1$

$\mu(A) = 1-(\mu(A^c)+\lambda\mu(A)\mu(A^c))$

Then

$\mu(A)+\mu(A^c)+\lambda\mu(A)\mu(A^c) = 1$

If $\lambda = 0$

Then $\mu(A) = 1-\mu(A^c)$

Proposition(2.4):

If $A_1, A_2 \in F$, $A_1 \subset A_2$

then $\mu(A_1) \leq \mu(A_2)$

Proof:

since $A_1 \subset A_2 \Rightarrow A_2^c \subset A_1^c$

$\Rightarrow \mu(A_2^c \cup A_2) = \mu(X) = 1$

$\mu(A_1^c) = \mu(A_2)+\lambda\mu(A_1^c)\mu(A_2)$

$$1-(\mu(A_1)+\lambda\mu(A_1)\mu(A_1^c)+\mu(A_2)+\lambda\mu(A_1^c)\mu(A_2)$$

$$1-\mu(A_1)[1+\lambda\mu(A_1^c)]+\mu(A_2)[1+\lambda\mu(A_1^c)]=1$$

$$[1+\lambda\mu(A_1^c)][\mu(A_2)-\mu(A_1)]=0$$

$$\mu(A_2)-\mu(A_1)\geq 0$$

$$\mu(A_2)\geq\mu(A_1)$$

Proposition (2.5) :- Let (X,F,μ) be a Sugeno fuzzy measure space then

$$\mu(A_1\cap A_2)=\frac{1-\mu(A_1\cup A_2)}{1+\lambda\mu(A_1\cup A_2)}$$

Proof:- since $A_1\cap A_2=(A_1\cup A_2)^c$

$$\mu(A_1\cap A_2)=\mu(A_1\cup A_2)^c=$$
$$1-\mu(A_1\cup A_2)-\lambda\mu(A_1\cup A_2)\mu(A_1\cup A_2)^c$$
$$\mu(A_1\cap A_2)=1-[\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)]$$
$$-\lambda[\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)]\mu(A_1\cap A_2)$$
$$\mu(A_1\cap A_2)=1-\mu(A_1)-\mu(A_2)-\lambda\mu(A_1)\mu(A_2)$$
$$-(\lambda\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2))\mu(A_1\cap A_2)$$
$$\mu(A_1\cap A_2)+(\lambda\mu(A_1)+\lambda\mu(A_2)+$$
$$\lambda^2\mu(A_1)\mu(A_2))\mu(A_1\cap A_2)$$
$$=1-\mu(A_1)-\mu(A_2)+\lambda\mu(A_1)\mu(A_2)$$
$$=\mu(A_1\cap A_2)[1+\lambda(\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)]$$
$$=1-(\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2))$$
$$\to\mu(A_1\cap A_2)=\frac{1-\mu(A_1\cup A_2)}{1+\lambda\mu(A_1\cup A_2)}$$

Remark:- from Proposition (2.5) we have

$$\mu(A_1-A_2)=\mu(A_1\cap A_2^c)=\frac{1-\mu(A_1\cup A_2^c)}{1+\lambda\mu(A_1\cup A_2^c)}$$

2- Continunity of sugeno fuzzy measure space:

Let $\{A_n\}$ be a sequence of subset of X The set of all points which are belong to infinitely many sets of the sequence $\{A_n\}$ is called the upper limit(or limit superior) of $\{A_n\}$ and (in symbol A^*) defined by

$A^*=\lim_{n\to\infty}\sup A_n=\{x\in A_n:$ for infinitely many $n\}=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k=\lim_{n\to\infty}\bigcup_{k=n}^{\infty}A_k$

Thus, $x\in A_n$ iff for all n , then $x\in A_k$ for some $k\geq n$

the lower limit (or limit inferior) of $\{A_n\}$ defined by A_* is the set of all points which belong to almost all sets of the sequence $\{A_n\}$, and denoted by

$A_*=\lim_{n\to\infty}\inf A_n=\{x\in A_n:$ for all but finitely many $n\}=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k=\lim_{n\to\infty}\bigcap_{k=n}^{\infty}A_k$

Thus, $x\in A_n$ iff for some n , then $x\in A_k$ for all $k\geq n$.

A sequence $\{A_n\}$ of subset of a set X is said to be converge if

$\lim_{n\to\infty}\sup A_n=\lim_{n\to\infty}\inf A_n=A$ and A is said to be the limit of $\{A_n\}$ we write

$A=\lim_{n\to\infty}A_n$ or $A_n\to A$ [4].

A sequence $\{A_n\}$ of subset of a set Ω is said to be increasing if $A_n\subset A_{n+1}$ for $n=1,2,...$. And is said to be decreasing if $A_{n+1}\subset A_n$ for $n=1,2,...$.

A monotone sequence of sets is one which either increasing or decreasing.

If $\{A_n\}$ is an increasing sequence of subset of a set X and $\bigcup_{n=1}^{\infty} A_n = A$,we say that A_n an increasing sequence of a set with limit A, or that A_n increase to A, write $A_n \uparrow A$, also if $\{A_n\}$ is a decreasing sequence of subset of a set X and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n a decreasing sequence of a set with limit A, or that the A_n decrease to A, write $A_n \downarrow A$ [4].

Theorem (3.1)[4]:

Let $\{A_n\}$ be a sequence of subset of a set X and let $A \subset X$

(1) If $A_n \uparrow A$ then $A_n^c \downarrow A^c$

(2) If $A_n \downarrow A$ then $A_n^c \uparrow A^c$

Theorem(3.2):

Let (X, F, μ) be Sugeno fuzzy measure space and $A_n \in F$ then

(i) If $A_n \uparrow A$ then $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$

(ii) If $A_n \downarrow A$ then $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$

(iii) If $A_n \downarrow \phi$ then $\lim\limits_{n\to\infty} \mu(A_n) = 0$

(iv) If $A_n \uparrow X$ then $\lim\limits_{n\to\infty} \mu(A_n) = 1$

Proof:- (i) since $A_n \uparrow A$, $A_n \subset A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = A$

$$\mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(A_1) + \sum_{i=2}^{\infty} \mu(A_i \mid A_{i-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{i=2}^{\infty} \mu(A_i \mid A_{i-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{i=2}^{\infty} \mu(A_i) - \mu(A_{i-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} (\mu(A_n) - \mu(A_1))$$

$$= \lim_{n\to\infty} (\mu(A_n)$$

then $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$

(ii) $A_n \downarrow A \to A_n^c \uparrow A^c$

Then from (i) $\lim\limits_{n\to\infty} \mu(A_n^c) = \mu(A^c)$

$$\Rightarrow \lim_{n\to\infty} (1 - \mu(A_n) - \lambda\mu(A_n^c)\mu(A_n)$$
$$= 1 - \mu(A) - \lambda\mu(A)\mu(A^c)$$

$$\Rightarrow 1 - \lim_{n\to\infty} \mu(A_n) - \lambda\mu(A_n^c)\mu(A_n) =$$
$$1 - \mu(A) - \lambda\mu(A)\mu(A^c)$$

$$= -\lim_{n\to\infty} \mu(A_n) = -\mu(A)$$

$$\therefore \lim_{n\to\infty} \mu(A_n) = \mu(A)$$

(iii)from(ii) $A_n \downarrow \phi$ then $\lim\limits_{n\to\infty} \mu(A_n) = \mu(\phi)$

$$\lim_{n\to\infty} \mu(A_n) = 0$$

(iv) $A_n \uparrow X$ then from (ii) $\lim\limits_{n\to\infty} \mu(A_n) = \mu(X)$

$$\to \lim_{n\to\infty} \mu(A_n) = 1 .$$

Theorem(3.3):- Let (X,F,μ) be Sugeno Fuzzy measure space and
$A_n \in F$, $\mu(A_n) \to 0$ as $n \to \infty$ then

$$\mu(A \cup A_n) = \lim_{n\to\infty} \mu(A - A_n) = \mu(A).$$

Proof:-
$A \subset A \cup A_n \to \mu(A) \le \mu(A \cup A_n) \qquad \forall n$

$$\mu(A \cup A_n) = \mu(A) + \mu(A_n) + \lambda\mu(A)\mu(A_n)$$

Since $\mu(A_n) \to 0$ then $\mu(A \cup A_n) \to \mu(A)$
and $A - A_n \subset A \subset (A - A_n) \cup A_n$

We have
$$\mu(A - A_n) \le \mu(A)$$
$$= \mu(A - A_n) + \mu(A_n) + \lambda\mu(A - A_n)\mu(A_n)$$

Since $\mu(A_n) \to 0$ we have

$$\mu(A - A_n) \to \mu(A).$$

3-pesduometric generating properties:

In this section we will study pesduometric generating properties in sugeno measure many authors was study with another measure as [3],[5].

Definition(4.1)[4]:

Let β be a collection of all real valued function defined on a set X, and $f, f_n \in \beta$, $n \in N$ and $A \in X$ we say that $\{f_n\}$ uniformly convergent to f on *A*, if for every $\varepsilon > 0$ there is $k \in Z^+$ such that $|f_n(x) - f(x)| < 0 \quad \forall n > k$ and $x \in A$, we write $f_n \to f$ on *A*.

Definition (4.2):

A set function μ is said to be have pseudometic generating property (p.g.p) if for any $\varepsilon > 0$, there exist $\gamma > 0$ such for any Borel sets *A* and *B*, $\mu(A) \vee \mu(B) < \gamma \Rightarrow \mu(A \cup B) < \varepsilon$.

Theorem(4.3):

A set function μ has p.g.p if and only if there exist a sequence $\{\gamma_n\}_n$ of real number such that $\gamma_n \downarrow 0$ and, for any sequence $\{A_n\}_n$ with $\mu(A_n) < \gamma_n$, the following inequalities hold

$$\mu\Big(\bigcup_{k=n+1}^{+\infty} A_k\Big) \le \gamma_n, \qquad n \ge 1.$$

Proof:

Let a set function μ has p.g.p then there exist $\gamma_1 \in (0, \frac{1}{2})$ such that

$\mu(A) \vee \mu(B) < \gamma_1$ implies $\mu(A \cup B) < \frac{1}{2}$

For above γ_1 there exist $\gamma_2 \in (0, \frac{1}{2^2} \wedge \gamma_1)$ to satisfy that $\mu(A) \vee \mu(B) < \gamma_2$ implies $\mu(A \cup B) < \gamma_1$ and, similarly there exist

$\gamma_3 \in (0, \frac{1}{2^3} \wedge \gamma_2)$ to satisfy that $\mu(A) \vee \mu(B) < \gamma_3$ implies $\mu(A \cup B) < \gamma_2$

Repeating this procedure,we can obtain a sequence $\{\gamma_n\}_n$ such that

$$0 < \gamma_{n+1} < \frac{1}{2^{n+1}} \wedge \gamma_n \qquad \forall n \geq 1 .$$

If $\mu(A_n) < \gamma_n, \quad \forall n \geq 1$

then we have $\mu(\bigcup_{k=n+1}^{n+r} A_k) \leq \gamma_n, \qquad n \geq 1$

So that $\theta(\bigcup_{k=n+1}^{+\infty} A_k) \leq \gamma_n, \qquad n \geq 1$

Conversely, for any $\varepsilon > 0$, there exist $n_0 \geq 1$ such that $\gamma_{n_o} < \varepsilon$

if we choose $\gamma = \gamma_{n_o+2}$ and , when $\mu(A) \vee \mu(B) < \gamma$ then we have

$$\theta(A \cup B) = \theta(\bigcup_{k=n_0+1}^{+\infty} A_n) \leq \gamma_{n_o} < \varepsilon$$

If we choose $\gamma = \gamma_{n_o+2}$ and when $\mu(A) \vee \mu(B) < \gamma$ then we have

$\mu(A \cup B) = \mu(\bigcup_{k=n+1}^{+\infty} A_k) \leq \gamma_n < \varepsilon$ where $A_{n_o+1} = A, \quad A_{n_o+2} = B,$ and otherwise $A_n = \phi$

Thus a set function μ has p.g.p.

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