Hyers-Ulam Stability of Second-Order Linear Differential Equations with Boundary Conditions

Qusuay H. Alqifiary and Soon-Mo Jung

College of Mathematics, University of Belgrade, Belgrade, Serbia
Work: University of Al-Qadisiyah, Al-Diwaniya, Iraq
E-mail: qhaq2010@gmail.com

Mathematics Section, College of Science and Technology, Hongik University, 339–701 Sejong, Republic of Korea
E-mail: smjung@hongik.ac.kr

Abstract

In this paper, we establish the Hyers-Ulam stability of linear differential equations of second order with boundary conditions.

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1 Introduction

In 1940, Ulam [24] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers [5] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X_1 \to X_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \quad (x, y \in X_1),$$

there exists a unique additive function $A : X_1 \to X_2$ with the property

$$\|f(x) - A(x)\| \le \varepsilon \quad (x \in X_1).$$

After Hyers's result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions (see [3, 6, 10, 18]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\varphi\big(f, y, y', \ldots, y^{(n)}\big) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left|\varphi\big(f, y, y', \ldots, y^{(n)}\big)\right| \le \varepsilon,$$

there exists a solution y_0 of the differential equation such that $|y(t) - y_0(t)| \leq K(\varepsilon)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [14, 15]). Thereafter, Alsina and Ger published their paper [1], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t)-y(t)| \leq \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leq 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [12, 22, 23] and in [13], respectively. Furthermore, Jung [7, 8, 9] has also proved the Hyers-Ulam stability of linear differential equations (see also [4]). Rus investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [20, 21]). Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [11, 25]). The results given in [8, 11, 12] have been generalized by Cimpean and Popa [2] and by Popa and Raşa [16, 17] for the linear differential equations of nth order with constant coefficients. Furthermore, the Laplace transform method was recently applied to the proof of the Hyers-Ulam stability of linear differential equations (see [19]).

In this paper, we prove the Hyers-Ulam stability of the second-order linear differential equations (3.1), (3.5), and (3.17) with boundary conditions.

2 Preliminaries

Lemma 2.1 *Let $I = [a, b]$ be a closed interval with $-\infty < a < b < \infty$. If $y \in C^2(I, \mathbb{R})$ and $y(a) = 0 = y(b)$, then*

$$\max_{x \in I} |y(x)| \leq \frac{(b-a)^2}{8} \max_{x \in I} |y''(x)|.$$

Proof. Let $M := \max_{x \in I} |y(x)|$. Since $y(a) = 0 = y(b)$, there exists $x_0 \in (a, b)$ such that $|y(x_0)| = M$. By the Taylor's theorem, we have

$$y(a) = y(x_0) + y'(x_0)(a - x_0) + \frac{y''(\xi)}{2}(a - x_0)^2,$$
$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(\eta)}{2}(b - x_0)^2$$

for some $\xi, \eta \in [a, b]$. Since $y(a) = y(b) = 0$ and $y'(x_0) = 0$, we get

$$|y''(\xi)| = \frac{2M}{(a - x_0)^2}, \quad |y''(\eta)| = \frac{2M}{(b - x_0)^2}.$$

If $x_0 \in \big(a, (a + b)/2\big]$, then we have

$$\frac{2M}{(a - x_0)^2} \geq \frac{2M}{\left(\frac{b-a}{2}\right)^2} = \frac{8M}{(b - a)^2}.$$

If $x_0 \in \big[(a+b)/2, b\big)$, then we have

$$\frac{2M}{(b-x_0)^2} \geq \frac{2M}{\left(\frac{b-a}{2}\right)^2} = \frac{8M}{(b-a)^2}.$$

Hence, we obtain

$$\max_{x\in I} |y''(x)| \geq \frac{8M}{(b-a)^2} = \frac{8}{(b-a)^2} \max_{x\in I} |y(x)|.$$

Therefore,

$$\max_{x\in I} |y(x)| \leq \frac{(b-a)^2}{8} \max_{x\in I} |y''(x)|,$$

which ends the proof. $\square$

Lemma 2.2 *Let $I = [a, b]$ be a closed interval with $-\infty < a < b < \infty$. If $y \in C^2(I, \mathbb{R})$ and $y(a) = 0 = y'(a)$, then*

$$\max_{x\in I} |y(x)| \leq \frac{(b-a)^2}{2} \max_{x\in I} |y''(x)|.$$

Proof. By the Taylor's theorem, we have

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(\xi)}{2}(x-a)^2$$

for some $\xi \in [a, b]$. Since $y(a) = y'(a) = 0$ and $(x-a)^2 \leq (b-a)^2$, we get

$$|y(x)| \leq \frac{|y''(\xi)|}{2}(b-a)^2$$

for any $x \in I$. Thus, we obtain

$$\max_{x\in I} |y(x)| \leq \frac{(b-a)^2}{2} \max_{x\in I} |y''(x)|,$$

which completes the proof. $\square$

3 Main results

In the following theorems, we prove the Hyers-Ulam stability of the following linear differential equation

$$y''(x) + \beta(x)y(x) = 0 \tag{3.1}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{3.2}$$

or with initial conditions

$$y(a) = 0 = y'(a) \tag{3.3}$$

where $I = [a, b]$, $y \in C^2(I, \mathbb{R})$, $\beta \in C(I, \mathbb{R})$, and $-\infty < a < b < \infty$.

Theorem 3.1 *Given a closed interval $I = [a, b]$, let $\beta \in C(I, \mathbb{R})$ be a function satisfying $\max\limits_{x \in I} |\beta(x)| < 8/(b - a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality*

$$|y''(x) + \beta(x)y(x)| \leq \varepsilon, \tag{3.4}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, as well as the boundary conditions in (3.2), then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation (3.1) with the boundary conditions in (3.2) such that

$$|y(x) - y_0(x)| \leq K\varepsilon$$

for any $x \in I$.

Proof. By Lemma 2.1, we have

$$\max_{x \in I} |y(x)| \leq \frac{(b-a)^2}{8} \max_{x \in I} |y''(x)|.$$

Thus, it follows from (3.4) that

$$\begin{aligned} \max_{x \in I} |y(x)| &\leq \frac{(b-a)^2}{8} \max_{x \in I} |y''(x) + \beta(x)y(x)| + \frac{(b-a)^2}{8} \max_{x \in I} |\beta(x)| \max_{x \in I} |y(x)| \\ &\leq \frac{(b-a)^2}{8} \varepsilon + \frac{(b-a)^2}{8} \max_{x \in I} |\beta(x)| \max_{x \in I} |y(x)|. \end{aligned}$$

Let $C := \frac{(b-a)^2}{8}$ and $K := \frac{C}{1 - C \max |\beta(x)|}$. Obviously, $y_0 \equiv 0$ is a solution of (3.1) with the boundary conditions in (3.2) and

$$|y(x) - y_0(x)| \leq K\varepsilon$$

for any $x \in I$. □

Theorem 3.2 *Given a closed interval $I = [a, b]$, let $\beta : I \to \mathbb{R}$ be a function satisfying $\max\limits_{x \in I} |\beta(x)| < 2/(b - a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality (3.4) for all $x \in I$ and for some $\varepsilon \geq 0$ as well as the initial conditions in (3.3), then there exist a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation (3.1) with the initial conditions in (3.3) and a constant $K > 0$ such that*

$$|y(x) - y_0(x)| \leq K\varepsilon$$

for any $x \in I$.

Proof. On account of Lemma 2.2, we have

$$\max_{x\in I}|y(x)| \le \frac{(b-a)^2}{2}\max_{x\in I}|y''(x)|.$$

Thus, it follows from (3.4) that

$$\begin{aligned}\max_{x\in I}|y(x)| &\le \frac{(b-a)^2}{2}\max_{x\in I}|y''(x)+\beta(x)y(x)| + \frac{(b-a)^2}{2}\max_{x\in I}|\beta(x)|\max_{x\in I}|y(x)| \\ &\le \frac{(b-a)^2}{2}\varepsilon + \frac{(b-a)^2}{2}\max_{x\in I}|\beta(x)|\max_{x\in I}|y(x)|.\end{aligned}$$

Let $C := \frac{(b-a)^2}{2}$ and $K := \frac{C}{1-C\max|\beta(x)|}$. Obviously, $y_0 \equiv 0$ is a solution of (3.1) with the initial conditions in (3.3) and

$$|y(x) - y_0(x)| \le K\varepsilon$$

for all $x \in I$. □

In the following theorems, we investigate the Hyers-Ulam stability of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{3.5}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{3.6}$$

or with initial conditions

$$y(a) = 0 = y'(a) \tag{3.7}$$

where $y \in C^2(I,\mathbb{R})$, $p \in C^1(I,\mathbb{R})$, $q \in C(I,\mathbb{R})$, and $I = [a,b]$ with $-\infty < a < b < \infty$.

Let us define a function $\beta : I \to \mathbb{R}$ by

$$\beta(x) := q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2$$

for all $x \in I$.

Theorem 3.3 *Assume that there exists a constant $L \ge 0$ with*

$$-L \le \int_a^x p(\tau)d\tau \le L \tag{3.8}$$

for any $x \in I$ and $\max_{x\in I}|\beta(x)| < 8/(b-a)^2$. If a function $y \in C^2(I,\mathbb{R})$ satisfies the inequality

$$|y''(x) + p(x)y'(x) + q(x)y(x)| \le \varepsilon \tag{3.9}$$

for all $x \in I$ *and for some* $\varepsilon \geq 0$ *as well as the boundary conditions in* (3.6)*, then there exist a constant* $K > 0$ *and a solution* $y_0 \in C^2(I, \mathbb{R})$ *of the differential equation* (3.5) *with the boundary conditions in* (3.6) *such that*

$$|y(x) - y_0(x)| \leq Ke^L\varepsilon$$

for any $x \in I$*.*

Proof. Suppose $y \in C^2(I, \mathbb{R})$ satisfies the inequality (3.9) for all $x \in I$. Let us define

$$u(x) := y''(x) + p(x)y'(x) + q(x)y(x), \tag{3.10}$$

$$z(x) := y(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right) \tag{3.11}$$

for all $x \in I$. By (3.10) and (3.11), we obtain

$$z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x) = u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)$$

for all $x \in I$.

Now, it follows from (3.8) and (3.9) that

$$\left|z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x)\right| = \left|u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right| \leq \varepsilon e^{L/2},$$

that is,

$$|z''(x) + \beta(x)z(x)| \leq \varepsilon e^{L/2}$$

for any $x \in I$. Moreover, it follows from (3.11) that

$$z(a) = 0 = z(b).$$

In view of Theorem 3.1, there exists a constant $K > 0$ and a function $z_0 \in C^2(I, \mathbb{R})$ such that

$$z_0''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z_0(x) = 0, \tag{3.12}$$

$$z_0(a) = 0 = z_0(b)$$

and

$$|z(x) - z_0(x)| \leq K\varepsilon e^{L/2} \tag{3.13}$$

for all $x \in I$.

We now set

$$y_0(x) := z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right). \tag{3.14}$$

Then, since

$$y_0'(x) = z_0'(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - \frac{1}{2}p(x)z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big), \tag{3.15}$$

$$\begin{aligned} y_0''(x) &= z_0''(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - p(x)z_0'(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ &\quad -\frac{1}{2}p'(x)z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) + \frac{1}{4}p(x)^2 z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big), \end{aligned} \tag{3.16}$$

it follows from (3.12), (3.14), (3.15), and (3.16) that

$$\begin{aligned} & y_0''(x) + p(x)y_0'(x) + q(x)y_0(x) \\ &= \Big(z_0''(x) + \Big(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\Big)z_0(x)\Big)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ &= 0 \end{aligned}$$

for all $x \in I$. Hence, y_0 satisfies (3.5) and the boundary conditions in (3.6).

Finally, it follows from (3.8) and (3.13) that

$$\begin{aligned} |y(x) - y_0(x)| &= \Big|z(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\Big| \\ &= |z(x) - z_0(x)|\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ &\le K\varepsilon e^{L/2}\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ &\le Ke^L\varepsilon \end{aligned}$$

for all $x \in I$. □

Theorem 3.4 *Assume that there exists a constant $L \ge 0$ such that (3.8) holds for all $x \in I$. Assume moreover that $\max_{x\in I}|\beta(x)| < 2/(b-a)^2$. If a function $y \in C^2(I,\mathbb{R})$ satisfies the inequality (3.9) for all $x \in I$ and for some $\varepsilon \ge 0$ as well as the initial conditions in (3.7), then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I,\mathbb{R})$ of the differential equation (3.5) with the initial conditions in (3.7) such that*

$$|y(x) - y_0(x)| \le Ke^L\varepsilon$$

for any $x \in I$.

Proof. Suppose $y \in C^2(I,\mathbb{R})$ satisfies the inequality (3.9) for any $x \in I$. Let us define $u(x)$ and $z(x)$ as in (3.10) and (3.11), respectively. By (3.10) and (3.11), we obtain

$$z''(x) + \Big(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\Big)z(x) = u(x)\exp\Big(\frac{1}{2}\int_a^x p(\tau)d\tau\Big)$$

for all $x \in I$.

Now, it follows from (3.8) and (3.9) that

$$\left|z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x)\right| = \left|u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right| \leq \varepsilon e^{L/2},$$

that is,

$$|z''(x) + \beta(x)z(x)| \leq \varepsilon e^{L/2}$$

for all $x \in I$. Furthermore, in view of (3.11), we have

$$z(a) = 0 = z'(a).$$

By Theorem 3.2, there exists a constant $K > 0$ and a function $z_0 \in C^2(I, \mathbb{R})$ such that

$$z_0''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z_0(x) = 0,$$
$$z_0(a) = 0 = z_0'(a)$$

and

$$|z(x) - z_0(x)| \leq K\varepsilon e^{L/2}$$

for any $x \in I$.

We now set

$$y_0(x) := z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right).$$

Moreover, since

$$y_0'(x) = z_0'(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right) - \frac{1}{2}p(x)z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)$$

and

$$\begin{aligned} y_0''(x) = {} & z_0''(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right) - p(x)z_0'(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right) \\ & - \frac{1}{2}p'(x)z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right) + \frac{1}{4}p(x)^2z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right), \end{aligned}$$

we have

$$\begin{aligned} & y_0''(x) + p(x)y_0'(x) + q(x)y_0(x) \\ & = \left(z_0''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z_0(x)\right)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right) \\ & = 0 \end{aligned}$$

for any $x \in I$. Hence, y_0 satisfies (3.5) along with the initial conditions in (3.7).

Finally, it follows that

$$\begin{aligned}
|y(x) - y_0(x)| &= \left| z(x) \exp\left(-\frac{1}{2} \int_a^x p(\tau) d\tau \right) - z_0(x) \exp\left(-\frac{1}{2} \int_a^x p(\tau) d\tau \right) \right| \\
&= |z(x) - z_0(x)| \exp\left(-\frac{1}{2} \int_a^x p(\tau) d\tau \right) \\
&\le K\varepsilon e^{L/2} \exp\left(-\frac{1}{2} \int_a^x p(\tau) d\tau \right) \\
&\le K e^L \varepsilon
\end{aligned}$$

for all $x \in I$. □

In a similar way, we investigate the Hyers-Ulam stability of the differential equation

$$y''(x) + \frac{k'(x)}{k(x)} y'(x) + \frac{l(x)}{k(x)} y(x) = 0 \tag{3.17}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{3.18}$$

or with initial conditions

$$y(a) = 0 = y'(a) \tag{3.19}$$

where $y \in C^2(I, \mathbb{R})$, $k \in C^1(I, \mathbb{R}\backslash\{0\})$, $l \in C(I, \mathbb{R})$, and $-\infty < a < b < \infty$.

Given a closed interval $I = [a, b]$, we set

$$\beta(x) := \frac{l(x)}{k(x)} - \frac{1}{2} \frac{d}{dx} \frac{k'(x)}{k(x)} - \frac{1}{4} \left(\frac{k'(x)}{k(x)} \right)^2$$

for all $x \in I$.

Theorem 3.5 *Assume that there exists a constant $L \ge 0$ with*

$$-L \le \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \le L \tag{3.20}$$

for any $x \in I$ and $\max_{x \in I} |\beta(x)| < 8/(b-a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality

$$\left| y''(x) + \frac{k'(x)}{k(x)} y'(x) + \frac{l(x)}{k(x)} y(x) \right| \le \varepsilon, \tag{3.21}$$

for all $x \in I$ and some $\varepsilon \ge 0$, as well as the boundary conditions in (3.18)*, then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation* (3.17) *with the boundary conditions in* (3.18) *such that*

$$|y(x) - y_0(x)| \le K e^L \varepsilon$$

for any $x \in I$.

Proof. Suppose $y \in C^2(I, \mathbb{R})$ satisfies (3.21) for all $x \in I$. Let us define

$$u(x) := y''(x) + \frac{k'(x)}{k(x)} y'(x) + \frac{l(x)}{k(x)} y(x), \tag{3.22}$$

$$z(x) := y(x) \exp\left(\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right) \tag{3.23}$$

for all $x \in I$. By (3.22) and (3.23), we obtain

$$z''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2} \frac{d}{dx} \frac{k'(x)}{k(x)} - \frac{1}{4} \left(\frac{k'(x)}{k(x)} \right)^2 \right) z(x) = u(x) \exp\left(\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right).$$

Further, it follows from (3.20) and (3.21) that

$$\begin{aligned} \left| z''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2} \frac{d}{dx} \frac{k'(x)}{k(x)} - \frac{1}{4} \left(\frac{k'(x)}{k(x)} \right)^2 \right) z(x) \right| &= \left| u(x) \exp\left(\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right) \right| \\ &\le \varepsilon \exp\left(\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right) \\ &\le \varepsilon e^{L/2}, \end{aligned}$$

that is,

$$|z''(x) + \beta(x) z(x)| \le \varepsilon e^{L/2}$$

for all $x \in I$. Moreover, it follows from (3.18) and (3.23) that

$$z(a) = 0 = z(b).$$

By Theorem 3.1, there exists a constant $K > 0$ and a function $z_0 \in C^2(I, \mathbb{R})$ such that

$$z_0''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2} \frac{d}{dx} \frac{k'(x)}{k(x)} - \frac{1}{4} \left(\frac{k'(x)}{k(x)} \right)^2 \right) z_0(x) = 0,$$
$$z_0(a) = 0 = z_0(b)$$

and

$$|z(x) - z_0(x)| \le K \varepsilon e^{L/2}$$

for any $x \in I$.

We now set

$$y_0(x) := z_0(x) \exp\left(-\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right).$$

Then, since

$$y_0'(x) = z_0'(x) \exp\left(-\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right) - \frac{1}{2} \frac{k'(x)}{k(x)} z_0(x) \exp\left(-\frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \right)$$

and

$$y_0''(x) = z_0''(x) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg) - \frac{k'(x)}{k(x)} z_0'(x) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg)$$
$$- \frac{1}{2} \bigg(\frac{k'(x)}{k(x)} \bigg)' z_0(x) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg)$$
$$+ \frac{1}{4} \bigg(\frac{k'(x)}{k(x)} \bigg)^2 z_0(x) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg),$$

we have

$$\begin{aligned} & y_0''(x) + \frac{k'(x)}{k(x)} y_0'(x) + \frac{l(x)}{k(x)} y_0(x) \\ &= \Bigg(z_0''(x) + \bigg(\frac{l(x)}{k(x)} - \frac{1}{2} \bigg(\frac{k'(x)}{k(x)} \bigg)' - \frac{1}{4} \bigg(\frac{k'(x)}{k(x)} \bigg)^2 \bigg) z_0(x) \Bigg) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg) \\ &= 0. \end{aligned}$$

Hence, y_0 satisfies (3.17) along with the boundary conditions in (3.18).

Finally, it follows that

$$\begin{aligned} |y(x) - y_0(x)| &= \bigg| z(x) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg) - z_0(x) \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg) \bigg| \\ &= |z(x) - z_0(x)| \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg) \\ &\le K\varepsilon e^{L/2} \exp\bigg(- \frac{1}{2} \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \bigg) \\ &\le K e^L \varepsilon \end{aligned}$$

for all $x \in I$. □

By a similar method as we applied to the proof of Theorem 3.4, we can prove the following theorem. Hence, we omit the proof.

Theorem 3.6 *Assume that* $\max_{x \in I} |\beta(x)| < 2/(b-a)^2$ *and there exists a constant* $L \ge 0$ *for which the inequality* (3.20) *holds for all* $x \in I$. *If a function* $y \in C^2(I, \mathbb{R})$ *satisfies the inequality* (3.21) *for all* $x \in I$ *and for some* $\varepsilon \ge 0$ *as well as the boundary conditions in* (3.19), *then there exist a constant* $K > 0$ *and a solution* $y_0 \in C^2(I, \mathbb{R})$ *of the differential equation* (3.17) *with the boundary conditions in* (3.19) *such that*

$$|y(x) - y_0(x)| \le K e^L \varepsilon$$

for any $x \in I$.

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