بسم الله الرحمن الرحيم

Topological Transformation Group

Samir A.Gubair
Department of Mathematics
College of Computer Science and Mathematics
Dephartment
University of AL-Qadisiya
M.Sc. class

Abstract

In this paper ,we introduce the concept of topological transformation group and clearing the properties of this concept ,Also we give the concept of compact topological transformation groups and study the properties of it.

KeyWords **:** Topological group ,compact topological transformation groups,G-spaces.

Introduction:

In [4]P.J.Higgins introduced the concept of topological group . Throughout this paper , the spaces x is topological spaces. In [4] P.J.higgins , A coset set ${}^{G}/_{H}$ is topological group if H normal subgroup .In [1] Bredon .E.Glen ,(let X be Hausdorff G-space , where G is compact topological group then map $\pi: X \to {}^{X}/_{G}$ is a closed map.

topological transformation Group

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Definition 1.1[5] Topological group is aset **G** with two structures:

i) G is group with respect to . , $^{-1}$,e

ii) G is a topological space.

And multiplication map

μ:G×G→G and the inversion map ν:G→G are both continuous .

Example 1.2 Let G=R be the real number then R is agruop with usual addition + and has the usual metric topology So (R,+,T_u) is a topological group .

Definition 1.3 [5]: Let G be a topological group, and let X be a topological space. By an action of G on X we mean a continuous map

$$\emptyset : G \times X \rightarrow X$$

1) $\varnothing(e, x) = x$, for all $x \in X$.

2) $\emptyset(g_1, \emptyset(g_2, x)) = \emptyset(g_1 g_2, x), for\ all\ x \in X, g_1, g_2 \in G$

We call the triple (G,X,$\varnothing$) a topological transformation group, and we also express this same thing by simply saying that X is a G–space.

Remark 1.4 [5]

1)The difference between the left and right action is not trivial one (at least for non-abelian topological groups)

However there is aone to one correspondence between them as follows:

If $\emptyset$ is aleft action of G on X then

$\varnothing' : X \times G \rightarrow X$ defined by

$\varnothing'(x, g) = \varnothing(g^{-1}, x)$

is a right action of Gon X similarly for right action Thus for every left action there is aconjugate right action and vice-versa so every theorem that is true of left action has aconjugate theorem for right action Because of this ,We will usually use aleft action.

2)We shall use the notation gx to denote and so $\emptyset(g, x)$the conditions (iand ii)becomes

i) ex=x for all x$\in$ X.

ii) $g_1(g_2 x) = (g_1\ g_2)x \quad for\ all\ x \in X. g_1\ g_2 \in G.$

Example 1.5 : Let G be a topological group. Then G acts on itself by multiplication,

$\varnothing : G \times G \rightarrow G, (g, g') \mapsto gg'$,We have

1)$eg' = g', forall\ g' \in G$

2)$g_1(g_2 g') = (g_1 g_2)g', forall g_1, g_2 \in G, g' \in G$

3) $\emptyset$ is continuous

Here 1) just expresses a property of the identity element of G, and 2) just expresses the fact that the multiplication in G is associative.

SO (G,G,$\emptyset$) is topological transformation group.

Example 1.6 Let G=R be the real additive topological group and X=R then R is R –space by $\phi : \Re \times \Re \rightarrow \Re$, $\phi(r, t) = r + t.$ for all r,t $\in R.$

Sol:

1) $\emptyset(\boldsymbol{e},\boldsymbol{t}) = \boldsymbol{e} + \boldsymbol{t} = \boldsymbol{t}$

2) $\boldsymbol{\phi}(\boldsymbol{r_1},\boldsymbol{\phi}(\boldsymbol{r_2},\boldsymbol{t}) = \boldsymbol{\phi}(\boldsymbol{r_1},\boldsymbol{r_2}+\boldsymbol{t})$

$$= \boldsymbol{r_1} + \boldsymbol{r_2} + \boldsymbol{t} = (\boldsymbol{r_1} + \boldsymbol{r_2}) + \boldsymbol{t}$$

$$= \boldsymbol{\phi}(\boldsymbol{r_1} + \boldsymbol{r_2},\boldsymbol{t})$$

3) $\boldsymbol{\phi}$ **is continuous.**

$\therefore$ $\boldsymbol{R}$ **is R-space.**

Definition 1.7 [5] let $(G,X,\varnothing)$be a topological transformation groups :

1.The orbit of $x \in X$ is defined to be the set

$G_x = \{\varnothing(g,x) : g \in G\} \subset X.$

2.the stabilizer of elements in G that fix x and denoted by S_x .
i.e.
$S_x = \{g \in G / \phi(g,x) = x\} \subset G$

3.The kernel of the action ϕ is the set
$Ker\phi = \{g \epsilon G / \phi(g,x) = x \; for \; all \; x \epsilon X\}.$

Lemma 1.8 [5] :Let (G,X,ϕ) be topological transformation groups then :

i) S_x is a subgroup of G.
ii) $Ker\phi$ is anormal sub group of G .
iii) $Ker\phi = \bigcap_{x \in X} S_x$.

Proof: i) Let $g_1, g_2 \epsilon S_x$, then
$\phi(g_1g_2)x = \phi(g_1,\phi(g_2x) = \phi(g_1,x) = x$,
and hence $g_1g_2 \in S_x$.
Furthermore $e \in S_x$; and if $g \in S_x$ then $gx = x$, and therefore also
$\emptyset(g^{-1},x) = \emptyset\left(g^{-1},\emptyset(g,x)\right)$
$\emptyset(g^{-1},x) = \emptyset(g^{-1}g,x)$
$\quad = \emptyset(e,x) = x$
which shows that $g^{-1} \in S_x$.
There for S_x sub group of G.

ii)let $h \in G$

$h(ker\emptyset)h^{-1} \subset ker\phi$

Let $g \in Ker\phi$

$\therefore (hgh^{-1})_{(x)}$=$h(g(h^{-1}x)) = h(h^{-1}{}_{x})$

= $hh^{-1}(x) = x$

$\forall\, x \in X\,, h(\ Ker\phi)h^{-1} \subset Ker\phi$

$But\ Ker\phi\ \subset\ h\,Ker\phi\,h^{-1}$

$\therefore\quad Ker\phi\ =\ h\,Ker\phi\,h^{-1}$

Then $Ker\phi$ normal sub group of G .

iii)

$g \in\ Ker\phi\ \leftrightarrow \phi(g,x) = x\,.\ \forall x \in X$

$\leftrightarrow g \in\ S_x\ \forall x \in X$

$\leftrightarrow g \in \cap\, S_x$

Lemma 1.9[5] let X be a G-space and assume that X is T_1_speace Then

i) S_x Is a closed sub group of G.

ii) Then the kernel Ker$\emptyset$ of the action $\emptyset$ is a closed subgroup of G .

Proof: i)We already Know that S_x is a supgroup of G . We shall show that S_x is closed in G .consider the map

$$\emptyset_x : G \to X.\, g \mapsto gx$$

Then ϕ is continuous since

$$\emptyset_x : G \cong G \times \{x\} \to G \times X \to X$$

,and hence

$$S_{x=\phi^{-1}\{x\}}$$
$$Is\ closed\ in\ G\,.$$

ii) since Ker$\emptyset =\cap_{x\in X} S_x$ .and S_x is closed sub group of G ,for each $x \in X$, the claim follows.

Now note that Ker$\emptyset$ is in fact normal subgroup of G . Let $h \in G$. Then

$$h(ker\emptyset)h^{-1} \subset ker\phi,$$
for if $g \in ker\phi$,We have that
$$(hgh^{-1})(x) = h\big(g(h^{-1}x)\big) = h\big(h^{-1}(x)\big)$$
$$= (hh^{-1})(x) = x,$$
For all $x \in X$.Likewise $h^{-1}(ker\emptyset)h \subset ker\phi$,and therefore
$$ker\phi \subset h(ker\phi)h^{-1}$$
Thus
$$ker\phi = h(ker\phi)h^{-1}$$

for all h $\in$ G, which shows that Kerϕ is a normal subgroup of G.

Definition 1.10 [5] Let (G,X,$\emptyset$) be a topological transformation groups then an action $\emptyset$ of G on X is called :

i)**Transitive** if the orbit G_x =X for all x$\in X$.

ii)**Trivial** if $Ker\phi = G$.

iii)**Effective** if $Ker\phi$ ={e}.

iv)**Free** if $S_x = \{e\}$ for all x$\in X$.

Example 1.11 :Let G be topological group then G acts on itself by multiplication ,

$\emptyset: G \times G \to G$,$(g, \grave{g}) \mapsto g\grave{g}$.The action is free and transitive .

Sol: To proof ϕ $is\ free\ and\ transitive$

$\because S_x = \{g \in G \ / \ \phi(g, \grave{g}) = \grave{g}\ \}$

$= \{g \in G \ / \ (g\grave{g}) = \grave{g}\ \}$ ={e}

$\therefore \phi$ is free.

$*$Let g $\in G$, $G_g \subset G$

$\because$ g $\in G \Rightarrow g^{-1} \in G\ and\ \grave{g}\ (g^{-1}) \in G$

Since G group $\phi(gg^{-1}, \grave{g}) = gg^{-1}\ \grave{g} = \grave{g}$

$\grave{g} \in G_g \to G \subset G_g$

G=Gg for all g $\in G$

$\therefore \phi\ transitive$.

Example 1.12 Let G be a topological group. Then G acts on itself by conjugation,

$\emptyset : G \times G \to G, (g, \bar{g}) \mapsto g\bar{g}g^{-1}$.

i)the stabilizer is center lizer of h .

ii)$Ker\phi$ is the center of G .

Sol:

i)Let g $\in G$. $S_g = \{$ g $\in G \ / \ \phi(g\bar{g}) = \bar{g}\}$

={ g $\in G \ / \ g\bar{g}\ \grave{g}^{-1}g = g\bar{g}\}$={ g $\in G / g\bar{g}e = g\bar{g}\}$

$=\{ g \in G / g\bar{g} = g\bar{g} \}$=center lizer of $\bar{g}$

ii)$Ker\phi = \{ g \in G / \phi(g,\bar{g}) = \bar{g} \ \forall \bar{g} \in G \}$

$=\{ g \in G / g\bar{g} \ g^{-1} = \bar{g} \ \forall \bar{g} \in G \} = \{ g \in G / g\bar{g} \ g^{-1} g = \bar{g}\, g \ \forall \bar{g} \in G \}$
$=\{ g \in G / g\bar{g} = \bar{g} g \ \forall \bar{g} \in G \}$=center of G
Example 1.13: If H be sub group of G then G is H-space by right translation and this action is free.
Sol:
$\emptyset: G \times H \to G, \emptyset(g,h) = R_h(g) = gh$
1)$\emptyset(g,h) = R_e(g) = ge \ \forall g \in G.$
2)$\emptyset(\emptyset(g,h_1)h_2) = \emptyset(R_{h1}(g),h_2)$
$=\emptyset(gh_1,h_2) = R_{h2}(gh_1)$
$=gh_1h_2 = R_{h1h2}(g)$
$= \emptyset(g,h_1h_2)$
4) $\emptyset$ is continuous .(R_H homeomorphism)
$\therefore \phi$ action of H on G.
To prove ϕ is free. $S_g = \{h \epsilon H : \emptyset(\boldsymbol{g},\boldsymbol{h}) = g \ \forall \ g \in G\}$
$=\{ h \epsilon H : R_h(g) = g \ \forall \ g \in G\}$
$=\{ h \epsilon H : (gh) = g \ \forall \ g \in G\}$
$=\{ h \epsilon H : (h) = e \ \forall \ g \in G\}$ ={e}
$\therefore \phi \ is \ free.$

Remark 1.14 [5]
Let X be a G–space. We define a relation $\sim$ in X as follows:
$x_1 \sim x_2 \leftrightarrow$, there exists g ∈ G such that $\emptyset(g,x_1) = x_2$.
We claim that $\sim$ is an equivalence relation in X.
1) $\sim$ i s reflexive: We have x $\sim$ x, for every x ∈ X, since Ø (e ,x) = x.
2) $\sim$ is symmetric: Suppose that $x_1 \sim x_2$, then there exists g∈ G such that Ø $(g,x_1) = x_2$.
Then $x_1 = \emptyset(e,x_1) = \emptyset((g^{-1}g),x_1) = \emptyset(g^{-1}, \emptyset(g,x_1)) = \emptyset(g^{-1},x_2)$. Thus $\emptyset(g^{-1},x_2) = x_1$, which shows that $x_2 \sim x_1$.

3) $\sim$ is transitive: Suppose that $x_1 \sim x_2$ and $x_2 \sim x_3$. Then there exist g,$\grave{g}$ ∈ G such that Ø $(g,x_1) = x_2$

and $\emptyset(\grave{g}, x_2) = x_3$. Now $\emptyset((\grave{g}\, g),x_1) = \emptyset(\grave{g}, \emptyset(g,x_1)) = \emptyset(\grave{g},x_2) = x_3$, which **shows that** $x_1 \sim x_3$.

Thus$\sim$ is an equivalence relation in X, and we have that the equivalence class [x] of a point x ∈ X equals

[x] = {y ∈ X :x∼ y} = {y ∈ X: y = Ø(g, x), g ∈ G}=G_x

Thus the equivalence class of x under ∼ is exactly the orbit Gx of x. By X/G (or more accurately by G\X)
we denote the set of equivalence classes under ∼, that is X/G
denotes the set of orbits in X. We call X/G the orbit space of the G–space X. BY $\pi: X \to X/_G$, .x ↦ Gx,
we denote the natural projection onto the orbit space. We give X/G the quotient topology induced by π:X ⟶X/G.

Remark 1.15 [5] let X be a G-space then the law of action ∅ defines the following mapping :
i. A homeomorphism
$\varphi_g: X \to X$ defined by $\varphi_g(x) = \varphi(g, x)$ which has inverse is φ_{g-1}.
i.e. $\varphi_g{}^\circ\varphi_{g-1}$=$\varphi_g{}^\circ\varphi_{g-1} = I_x$.
Where I_x is the identity map on X .
ii. A continuous map $\varphi_x: G \to X$ defined by $\varphi_x(g) = \varphi(g, x)$
for each x∈ X .(since $\varphi_x: G \cong G \times \{x\} \subset G \times X \overset{\varphi}{\to} X$.)
Note that $S_x = \varphi_x^{-1}(\{x\})$ and $G_{x=}\varphi_x(G)$.
iii. open continuous surjection map $\pi: X \longrightarrow X/_G$
Sol: let V open in X we show that $\pi(V)$ open $in\ X/_G$.
To prove $\pi^{-1}(\pi(V)) open$ in X.
∵ $\pi^{-1}(\pi(V))$ =$\bigcup_{g \in G} gv$.
∵ v open in X ⟹ $\bigcup_{g \in G} gv$ open in X .
∴ $\pi^{-1}(\pi(V))$ open in X .
$(\pi(V))$ open in $X/_G$.
Definition 1.16 [1] let X be G- space then
1) we say that X is free G- space on X if the action of G on X is free.
2) we say that X is effective G- space on X is effective G- space if the action of G on X is effective.
3) We say that X is transitive G-space if the action of G a X is transitive .

Remark 1.17 [1] let (G,X,φ) be a topological transformation groups then :
1.If Homeo(x) represent the set of all homeomorphism on X which is group under the composition law of functions Then the map
$\acute{\varphi}$: G⟶ Homeo(x) ,g⟼ φ_g, is homomorphism of groups since
$\acute{\varphi}\ (g_1, g_2) = \varphi_{g1g2} = \varphi_{g1} o \varphi_{g2}$
$\grave{\varphi}(g_1) o\ \grave{\varphi}\ (g_2)$.

2.If H⊆G and A⊆ X we put HA=$\varphi(H \times A) = \{\varphi(h, a): h\epsilon H\ , a\epsilon A\}$ and A is called **invariant** under H if and only if HA⊆ A.

Definition 1.18 [5] Let (G,X, φ) be topological transformation group H be a supgroup of G and A ⊂ X. such that A is invariant under H and $\dot{\varphi}$ =$\varphi\ /_{H\times A}$ Then (H,A, $\dot{\varphi}$) is called sub topological transformation group .

Remark 1.19 [5] Let X be a G-Space then the law of action φ define the following mapping

$\vartheta: G \times X \to X \times X, \vartheta(g, x) = (x, \emptyset(g, x))$

i. The map ϑ is continuous since

$$\vartheta: G \times X \xrightarrow{I_G\times\Delta} G \times X \times X \xrightarrow{\varphi\times I_x} X \times X \cong X \times X$$

$$(g, x) \longmapsto (g, x, x) \longmapsto (\varphi(g, x\ ), x) \mapsto (x, \varphi(g, x))$$

ii. The image ϑ is the graph of the equivalence relation defined by the a action φ .

iii. If X is free G-Space then ϑ injective function .

Proof:

TO prove ϑ is injective

Let $\vartheta(g_1, x_1) = \vartheta(g_2, x_2)$

$$\big(x_1, \varphi(g_1, x_1)\big) = (x_2, \varphi(g_2, x_2))$$

$$\Longrightarrow x_1 = x_2\ and\ \varphi(g_1, x_1) = \varphi(g_2, x_2)$$

Let $x_1 = x_2 = x \Longrightarrow\ \varphi(g_1, x) = \varphi(g_2, x)$

Now $and\ \varphi(g_1^{-1}g_2, x) = \varphi(g_{12}, \varphi(g_2^{-1}, x)$

$$\varphi(e, x) = x\ \Longrightarrow g_1^{-1}g_2 \in s_x = \{e\}$$

$$\Longrightarrow g_1^{-1}g_2 = e \Longrightarrow g_1 = g_2$$

$$(g_1, x_1) = (g_2, x_2)$$

∴ ϑ is injective.

Theorem 1.20 [1] Let G be a compact topological group, and let φ:G×X⟶ X be an action of G on a Hausdorff space X Then

i) φ is a closed map.

ii) If A be a closed subset of X. Then GA = $\{ga/g \in G, a \in A\}$ is closed in X.

iii) If A is compact, then GA is compact.

Proof:
If A is a closed subset of X, then G × A is a closed subset of G × X. Hence GA = φ (G × A) is closed in X.
If A is compact, then G × A is compact, and hence
GA = φ (G × A) is compact.

Theorem 1.21 [1] Let X be a Hausdorff G–space, where G is a compact topological group.
Then:
1) The map $\pi: X \longrightarrow X/_G$ is a closed map
2) The orbit space X/G is Hausdorff.
3)the map $\pi: X \longrightarrow X/_G$ is compact map .
(If $B \subset X/_G$ is a compact sup set of $X/_G$ then $\pi^{-1} is\ compact$)
4)X is compact iff $X/_G$ is compact.

Proof :

1) $To\ prove\ \pi: X \longrightarrow {}^{X}/_{G}$ closed map

$\because A\ is\ \ closed\ in\ X\ \ to\ prove\ \pi(A) closed\ in\ {}^{X}/_{G}$

i.e.$\pi^{-1}\big(\pi(A)\big) closed\ in\ X$

$\because \pi^{-1}\big(\pi(A)\big) = \bigcup_{g\in G} gA = GA$

$\because GA\ is\ closed\ \ \Longrightarrow \pi(A) is\ closed\ in\ {}^{G}/_{X}$

$\therefore \pi\ is\ closed$

2) Let $\grave{x}, \grave{y}\ \in\ {}^{X}/_{G}\ \ni \grave{x}\ \neq\ \grave{y}$

Let x ,y $\in X$ such that $\pi(x) = \grave{x}, \pi(y) = \grave{y} then\ \pi^{-1}(\grave{x}) = G_X\ and\ \pi^{-1}(\grave{y}) = G_y$.
$the\ orbit\ G_x\ and\ G_y\ are\ compact\ and\ dis\ joint\ (i.e. G_x \bigcap G_y = \phi. \therefore \exists u, v\ open\ set\ in\ X \ni G_x \subseteq u, G_y \subseteq v. with\ \bar{u} \cap G_y = \phi. since\ \pi\ is\ closed\ map\ . then\ \pi(\bar{u}) is\ closed\ in\ {}^{X}/_{G}. thus\ {}^{X}/_{G} - \pi(\bar{u}) is\ open\ neghborhood\ of\ y\ in\ {}^{X}/_{G}. SO\ \pi(u) open\ in\ {}^{X}/_{G}\ \ni \bar{x} \in \pi(u). {}^{X}/_{G} - \pi(\bar{u}) open\ in\ {}^{X}/_{G}\ \ni$

$$\bar{y}\ \in \left({}^{X}/_{G} - \pi(\bar{u})\right). \quad and\ \ \pi(u) \cap$$

${}^{X}/_{G} - \pi(\bar{u}) = \phi. this\ completes\ the\ proof\ of (2)$.

Remark 1.22 [1] Let X be Hausdorff a G- Space ,where G is a compact topological group Then

i. The map$\varphi_x: G \longrightarrow X$ is closed map.

ii.each orbit is compact since $G_x = \varphi_x(G)$.

iii.The stabilizer S_x is closed since $S_x = \varphi_x^{-1}(\{x\})$ and {x}is closed in X AlsoS_x is compact .(closed sub space in compact space)

Proof :

i. Let A is closed in G .

$\because A$ closed and G is compact .

$\therefore A \; compact \; in \; G$.
$\therefore \varphi_x(A) compact \; in \; X$.
$\because \varphi_x(A) compact \; and \; X is \; Hausdorff$.
$\therefore \varphi_x(A) \; is \; closed$.

ii) $\varphi_x: G \longrightarrow X. \Longrightarrow \varphi_x(G) = \{\varphi_x(g): g \in G\} = orbit \; G = G_x$
$since \; G \; closed \; and \; \varphi_x \; closed \; map$.
$\varphi_x(G) = G_x \; is \; closed$.
$\therefore \varphi_x(G) compact$.

$iii. \varphi_x: G \longrightarrow X, \varphi_x$(g)=gx.$\varphi(g,x) is \; continuous \; and \; surjective$.
$\because X \; is \; H \; ausdorf \; . \Longrightarrow X \; is \; T_1 - space$.
$\therefore \{x\} is \; closed \; in \; X \; \forall \; x \in X$.
$\because \varphi \; is \; continuous$.
$\therefore \varphi_x^{-1}(\{x\}) = S_x \; is \; closed \; in \; G$.
$\because S_x \; closed \; in \; G \; and \; G \; is \; compact \; and \; \varphi_x \; continuous$.
$\therefore S_x \; compact$. ■

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